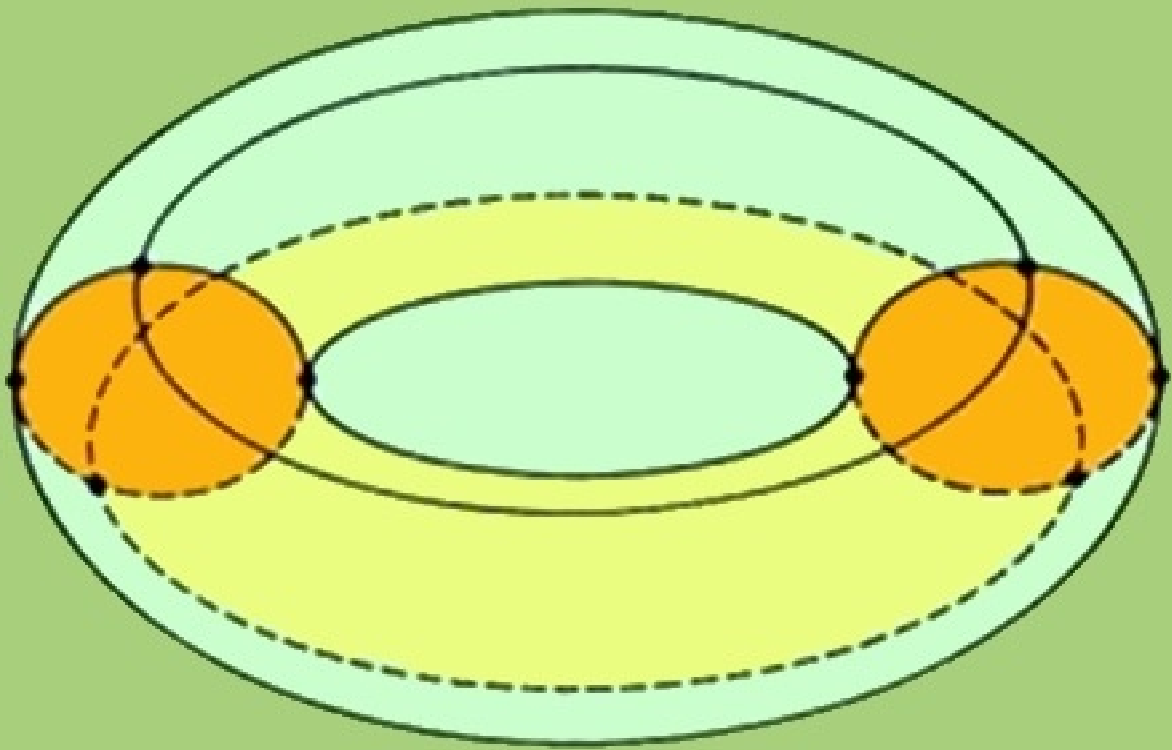
Michael Henle



A COMBINATORIAL INTRODUCTION TO TOPOLOGY

A Combinatorial Introduction to Topology

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Preface

Topology is remarkable for its contributions to the popular culture of mathematics. Euler's formula for polyhedra, the four color theorem, the Möbius strip, the Klein bottle, and the general notion of a rubber sheet geometry are all part of the folklore of current mathematics. The student in a first course in topology, however, must often wonder where all the Klein bottles went, for such courses are most often devoted to **point set topology,** the branch of topology that lies at the foundation of modern analysis but whose intersection with the popular notions of topology is almost empty. In contrast, the present work offers an introduction to **combinatorial** or **algebraic topology,** the other great branch of the subject and the source of most of its popular aspects.

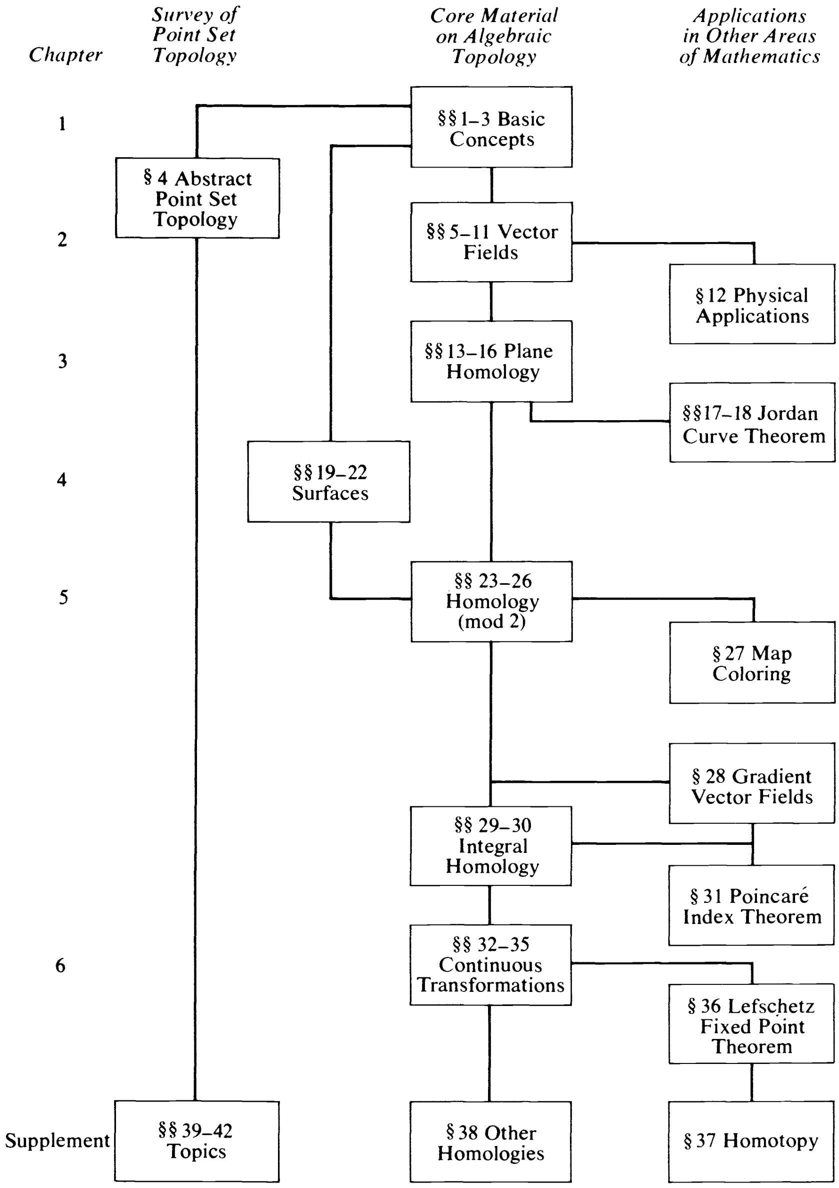
There are many good reasons for putting combinatorial topology on an equal footing with point set topology. One is the strong intuitive geometric appeal of combinatorial topology. Another is its wealth of applications, many of which result from connections with the theory of differential equations. Still another reason is its connection with abstract algebra via the theory of groups. Combinatorial topology is uniquely the subject where students of mathematics below graduate level can see the three major divisions of mathematics—analysis, geometry, and algebra—working together amicably on important problems.

In order to bring out these points, the subject matter of this introductory volume has been deliberately restricted. Most important is the emphasis on surfaces. The advantage is that the theorems can be easily visualized, encouraging geometric intuition. At the same time, this area is full of interesting applications arising from systems of differential equations. To bring out the interaction of geometry and algebra, it was decided to develop one algebraic tool in detail, homology, rather than several algebraic tools briefly. Thus an alternative title for the book could be *Homology of Surfaces*. These limitations of subject matter may be defended not only on the pedagogical grounds just outlined but also on historical grounds. Topological investigations of surfaces go back to Riemann and have always played an important role as a model for the study of more complicated objects, while homology, also one of the earliest areas of research, played a key role in the introduction of algebra into topology.

It is hoped that this book will prove useful to two classes of students: upper-level undergraduates as an introduction to topology, and first-year graduate students as a prelude to more abstract and technical texts. The prerequisites for reading this book are only some knowledge of differential equations and multivariate calculus. No previous acquaintance with topology or algebra is required. Point set topology and group theory are developed as they are needed. In addition, a supplement surveying point set topology is included both for the interested student and for the instructor who wishes to teach a mixture of point set and algebraic topology.

This book is suitable for a variety of courses. For undergraduates, a short course (8–10 weeks) can be based on the first four-and-a-half chapters (up to §28), while a longer course (12–15 weeks) could cover all the first five chapters as well as parts of Chapter Six (§§36,37, and 38 are all logical stopping points). The Interdependency Chart (opposite) reveals a number of sections that can be omitted if time grows short, although inclusion of as many as possible of these application sections is recommended. The supplement can be used to replace four sections of algebraic topology if desired. It is independent of Chapters Two through Six and so can be taken up anytime after Chapter One. A graduate class can cover the book in less than a semester, especially if point set topology is assumed. The remainder of the course can be devoted to a more technical treatment of algebraic topology or to further investigation of the topics in this book. Suggestions for further reading are given at the end of each chapter as well as in special sections at the end of the book.

INTERDEPENDENCY CHART



An important feature of the book is the problems. *They are an integral part of the book.* The argument of the text often depends upon statements made in the exercises. Therefore they should at least be read, if not worked out. A section of hints and answers (mostly hints) is provided at the end of the book.

The creation of algebraic topology is one of the triumphs of twentieth-century mathematics. The goal of this book is to show how geometric and algebraic ideas met and grew together into an important branch of mathematics in the recent past. At the same time, the attempt has been made to preserve some of the fun and adventure that is naturally part of a mathematical investigation.

I would like to thank Peter Renz and the rest of the staff at W. H. Freeman and Company for their encouragement and assistance. I particularly wish to thank Professors Victor Klee and Isaac Namioka for their careful reading of the manuscript. Finally, I wish to acknowledge the inspiration of Mrs. Helen Garstens, in whose eighth and ninth grade classes I first encountered topology.

Michael Henle
June 1977

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Basic Concepts

§1 THE COMBINATORIAL METHOD

Topology is a branch of geometry. To get an idea of its subject matter, imagine a geometric figure, such as a circular disk, cut from a sheet of rubber and subjected to all sorts of twisting, pulling, and stretching. Any deformation of this sort is permitted, provided the rubber can withstand it without ripping or tearing. Figure 1.1 shows some of the shapes the disk might assume. The rubber must be very flexible! Perhaps silly putty would work. Technically these distortions are called continuous transformations. The formal definition of **topology** is the study of properties of figures that endure when the figures are subjected to continuous transformations. Technical details will come later (§2). In this section we wish to describe some of the ideas of topology at the intuitive level where topology has earned the nickname "rubber sheet geometry."

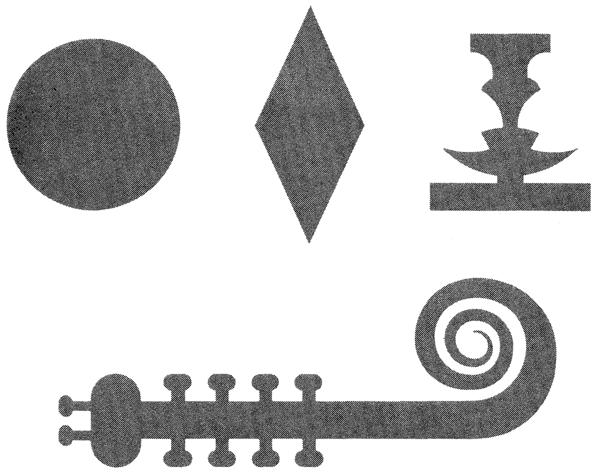


Figure 1.1 A disk and some topological equivalents.

From Figure 1.1 it is clear that a continuous transformation can destroy all the usual geometric properties of the disk, such as its shape, area, and perimeter. Therefore these are *not* topological properties of the disk. Any topological properties the disk has must be shared with the other shapes in Figure 1.1. These shapes are called topologically equivalent, meaning that any one can be continuously transformed to any other. Although topology will seem very different from Euclidean geometry, the two are fundamentally alike. In Euclidean geometry there is also a basic group of transformations, called congruences. The subject matter of Euclidean geometry is determined by the congruences, just as the subject matter of topology is determined by the continuous transformations. The congruences all preserve distances. For this reason they are also called rigid motions. Under a rigid motion a disk remains a circular disk with the same shape, radius, area, and perimeter. Therefore these are Euclidean properties of the disk. Euclidean geometry can be defined as the study of those properties of figures that endure when the figures are subjected to rigid motions.

What is a topological property of the disk? For one thing, the disk is in one piece. This is a topological property, since any attempt to divide the disk into more than one piece would require cutting or tearing, and these are forbidden in continuous transformations. Another topological property of the disk is that it has just one boundary curve. Another plane figure of some importance in topology, the **annulus** (ring shape) (see Figure 1.2), can be distinguished from the disk by this property, since the annulus has

Figure 1.2 An annulus and friend (another annulus).

two boundary curves. Another property distinguishing the disk from the annulus is that the latter divides the plane into two parts while the former does not.

Exercises

1. Examine some familiar objects topologically. For example, which of the following objects are topologically equivalent: hand iron, baseball bat, pretzel, telephone handset, rubber band, chair (consider several types), funnel, a scissor, Frisbee, etc.?
2. Why is it said that a topologist is someone who can't tell a doughnut from a coffee cup?

Let us call any figure topologically equivalent to a disk a **cell**. A cell is one of the simplest of topological figures. The first principle of combinatorial topology is to study the complicated figures that can be built in some way from simple figures. To put this principle into practice in this book, we will restrict ourselves to figures that can be constructed from cells by gluing and pasting them together along their edges. Figure 1.3 gives some examples. The edges of the cells sewn together to make each example are clearly marked. The numbers of cells used are 4, 7, 8, 8, 3, and 5, respectively. These numbers are not rigidly determined. For example, it would be simpler to build a sphere from just two cells, like the northern and southern hemispheres of a globe, or the two hourglass-shaped cells that are sewn together to make a baseball (Figure 1.4). Figures like these are called **complexes**. All but the most pathological of geometric surfaces are complexes; that is, they can be regarded as built in this way from cells.

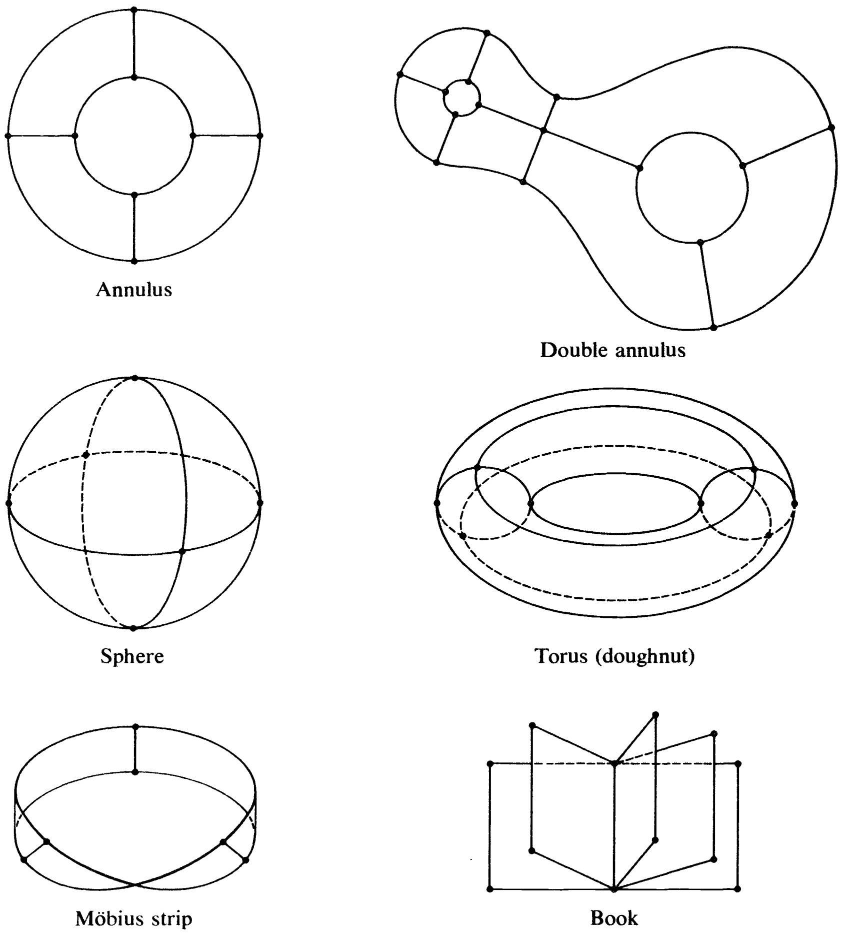


Figure 1.3 Some complexes.

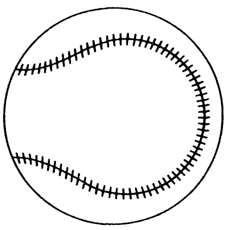


Figure 1.4 Another sphere.

Exercises

3. Examine the surfaces presented by some familiar objects. In each case try to show how the surface can be constructed from cells.

4. Adopt the rule that no cell is to be sewn to itself. Find the minimum number of cells needed to make an annulus, a double annulus, and so forth. Do these numbers change if cells are permitted to be sewn to themselves?

The combinatorial method is used not only to construct complicated figures from simple ones but also to deduce properties of the complicated from the simple. This idea is actually common to much of mathematics. In combinatorial topology it is remarkable that the only machinery needed to make these deductions is the elementary process of counting! This is the miracle of combinatorial topology, that counting alone suffices to found a sophisticated geometric theory. This reliance on counting is what distinguishes combinatorial topology from other branches of topology and geometry.

In order to count we must have something to count. To this end we jazz up the cells a bit. We call a cell a **polygon** when a finite number of points on the boundary are chosen as vertexes. The sections of boundary in between vertexes are then called **edges**. A polygon is called an ***n*-gon**, where n is the number of vertexes. The complexes in Figure 1.3 are actually built from cells in the form of polygons. Thus the annulus there is composed of four 4-gons, while the double annulus is composed of three 4-gons and four 5-gons. As a further example, Figure 1.5 shows a set of four 3-gons, a 4-gon, and a 5-gon that have often been successfully sold commercially as a puzzle. The same figure shows two complexes (among many) that can be made from these polygons: a rectangle and a letter T. Of course, considered topologically these two complexes are equivalent. In fact, after the pieces of the puzzle are sewn together, both complexes are cells.

The choice of certain points as vertexes in order to be able to regard cells as polygons may seem irrelevant to topology. However, it provides something to count and so affords the combinatorial method something with which to work. In the future we assume that all complexes are formed from polygons, and furthermore we adopt the rule that vertexes are sewn to vertexes and whole edges are sewn to whole edges.

As an application of the combinatorial method, here is a derivation of a famous result: Euler's formula for polyhedra. A **polyhedron** is a complex that

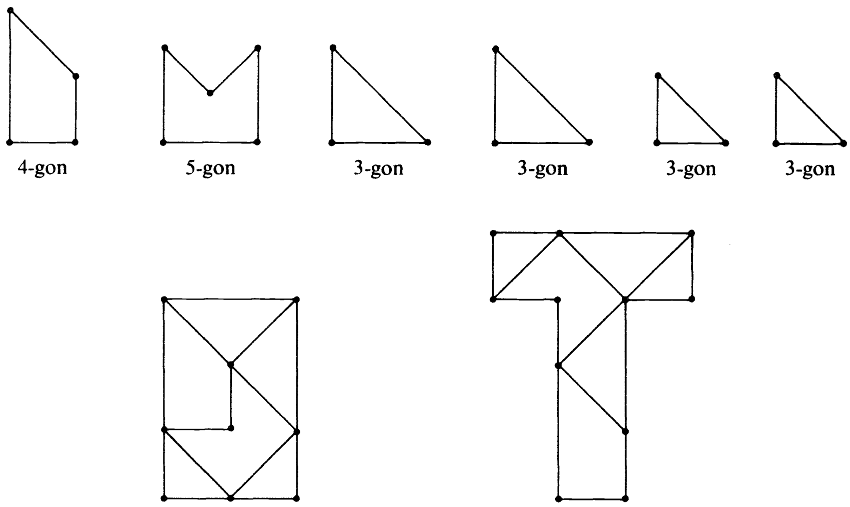


Figure 1.5 The T puzzle.

is topologically equivalent to a sphere. For example, the sphere in Figure 1.3 is a polyhedron, an **octahedron** in this terminology. Given a polyhedron, let F stand for the number of cells (called **faces** in this context), E the number of edges, and V the number of vertexes. Euler's formula states that

$$F - E + V = 2 \tag{1}$$

This remarkable result is the earliest discovery in combinatorial topology. It is due to Descartes (1639) but bears the name of Euler, who rediscovered it and published a proof (1751). The following proof is due to Cauchy (1811).

We begin by removing one of the faces of the polyhedron. The remainder is topologically equivalent to a cell and so may be flattened into a plane. For example, Figure 1.6 shows this operation being performed on a cube. The result is a cell in the plane divided into polygons. By removing one face and leaving edges and vertexes intact, the sum $F - E + V$ has been decreased by one. We must now prove in general for a complex equivalent to a cell that

$$F - E + V = 1 \tag{2}$$

This is called Euler's formula for a cell. It applies, for example, to the two complexes of Figure 1.5.

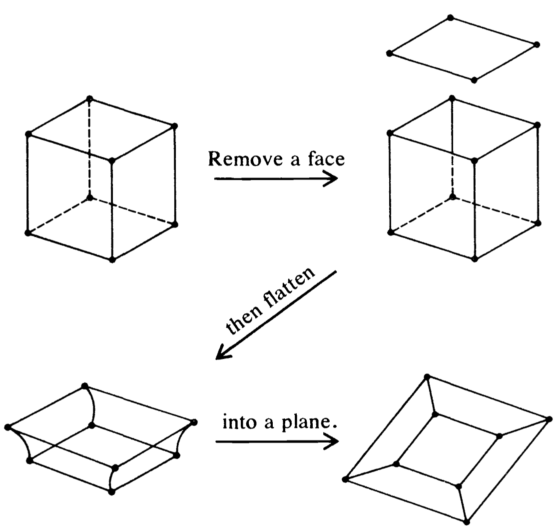


Figure 1.6 Proof of Euler's formula.
Step one: removing a face.

The next step is to triangulate the complex: divide each polygon into triangles by drawing diagonals, as in Figure 1.7. Each diagonal adds one edge and one face to the complex so that the quantity $F - E + V$ is unchanged by this process. In the final step, the triangles of the figure are removed one by one starting with those on the boundary. There are two types of removal, depending on whether the triangle being removed has one or two edges on the boundary (Figure 1.8). In the first case, the removal decreases both F and E by one, while in the second case, F and V decrease by one and E decreases by two. In any event, the quantity $F - E + V$ is

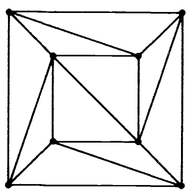


Figure 1.7 Step two:
the topless cube triangulated.

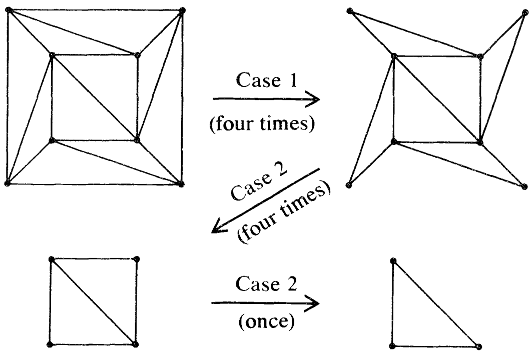


Figure 1.8 Step three: removal of triangles.

unchanged by the removal. Eventually we are left with just one triangle for which $F = 1$, $E = 3$, and $V = 3$. Obviously here $F - E + V = 1$. Since Steps 2 and 3 did not alter the sum $F - E + V$, this proves Euler's formula.

Exercises

5. Repeat Exercise 4, assuming this time that all cells are in the form of polygons and supposing that no cell can be sewn to itself or to any other cell along more than one edge.

6. Confirm Euler's formula for polyhedra in the cases of the cube and octahedron. Confirm Euler's formula for cells for the complexes in Figure 1.5.

7. Carry out the steps of the proof of Euler's formula for a tetrahedron (triangular pyramid) and an octahedron.

To appreciate the topological value of Euler's formula, consider it for a moment from the point of view of the sphere. Euler's formula says that no matter how the sphere may be divided into polygons, the sum of the number of faces minus the number of edges plus the number of vertexes is two: *always*. Rather like the surface area of the sphere, the number two is a property of the sphere itself, for although it is discovered only by dividing the sphere into polygons and then counting and adding faces, edges, and vertexes, the number two is independent of the manner of this division. Unlike surface area, however, the number two is also a *topological property* *of the sphere*, since the proof of Euler's formula is valid for any figure topologically equivalent to the sphere. Thus a topological property can result from the introduction of the seemingly irrelevant vertexes and edges. This is the whole point of the combinatorial method. Although its meaning is not completely clear yet, in fact *the number two is the single most important topological property of the sphere.* The remainder of this book may be regarded as an elaborate justification of this assertion. The number two is called the **Euler characteristic** of the sphere. Similarly, on account of (2), the Euler characteristic of the cell is one.

As an application of Euler's formula, we can determine all the regular polyhedra. A **regular** polyhedron is one in which all faces have the same number of edges, and the same number of faces meet at each vertex. The cube is an example of a regular polyhedron: all faces have four edges and all vertexes lie on three faces. The Greeks discovered five regular polyhedra. They are called the **Platonic solids**, and their construction is the climax of Euclid's Elements. We shall soon see why there are no others. Let a be the number of edges on each face of a regular polyhedron, and let b be the number of edges meeting at each vertex. Then the number aF counts all the edges face by face. Each edge is counted twice in this way, since each edge belongs to two faces. In other words, $aF = 2E$. Similarly, since each edge has two vertexes, $bV = 2E$. Euler's formula now reads

$$\frac{2E}{a} - E + \frac{2E}{b} = 2$$

or

$$\frac{1}{a} + \frac{1}{b} - \frac{1}{2} = \frac{1}{E} \tag{3}$$

This is an example of a **Diophantine equation**, so called because the solutions a, b, and E by the nature of the problem must be positive integers, and because equations with solutions of this type were studied by the Greek mathematician Diophantus. The restriction to the positive integers is an immense simplification. In this case it makes a complete solution possible. Actually both a and b must be greater than two, since faces or vertexes with only one edge are impossible, and those with just two edges are usually deliberately excluded. Since both a and b are greater than two, they must also be less than six in order that the left-hand side of (3) be positive. Thus when a and b are greater than two, there are only a finite number of possibilities, and the solutions may be found by trial and error. There turn out to be just five solutions, each corresponding to a polyhedron that can actually be constructed using Euclidean polygons. The solutions are illustrated in Figure 1.9.

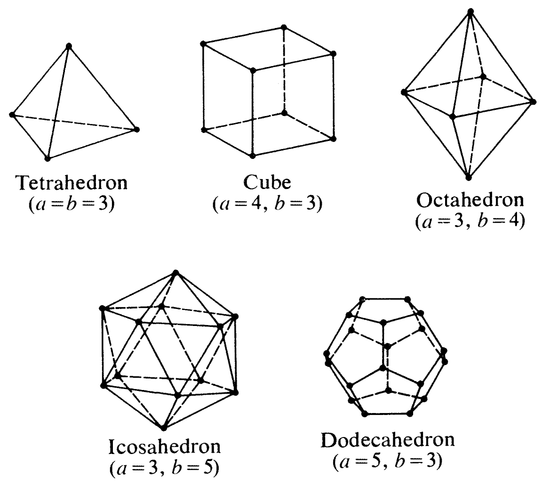


Figure 1.9 The platonic solids.

Exercises

8. Given a polyhedron (not necessarily regular), let F_n be the number of n-gon faces, and let V_n be the number of vertexes at which exactly n edges meet. Verify the following equations:

(a) $F_3 + F_4 + F_5 + \cdots = F$

(b) $V_3 + V_4 + V_5 + \cdots = V$

(c) $3F_3 + 4F_4 + 5F_5 + \cdots = 2E$

(d) $3V_3 + 4V_4 + 5V_5 + \cdots = 2E$

(e) $(2V_3 + 2V_4 + 2V_5 + \cdots) - (F_3 + 2F_4 + 3F_5 + \cdots) = 4$

(f) $(2F_3 + 2F_4 + 2F_5 + \cdots) - (V_3 + 2V_4 + 3V_5 + \cdots) = 4$

9. Using (e) and (f) of the preceding exercise, show that

$$(F_3 - F_5 - 2F_6 - 3F_7 - \cdots) + (V_3 - V_5 - 2V_6 - 3V_7 - \cdots) = 8$$

Conclude that every polyhedron must have either triangular faces or trivalent vertexes. Similarly, by eliminating F_6 between (e) and (f), deduce that

$$(3F_3 + 2F_4 + F_5 - F_7 - 2F_8 - \cdots) - (2V_4 + 4V_5 + 6V_6 + 8V_7 + \cdots) = 12$$

Therefore every polyhedron must have triangles, 4-gons, or 5-gons. Show that if a polyhedron has no triangles or 4-gons, then it must have at least twelve 5-gons. Is there a polyhedron with this minimal number of 5-gons?

10. F and V play symmetrical roles in Euler's formula. Let two polyhedra be called **dual** if the number of faces of one is the number of vertexes of the other and vice versa. Find the duals of the platonic solids.

11. Find the solutions of (3) when a or b are allowed to equal two. The corresponding polyhedra are called **degenerate**. Draw them.

12. Show that the Euler characteristic of the annulus is zero. Investigate the characteristic of the double annulus, torus, and so forth.

13. Little is known of the life of Diophantus, but the following problem from a Greek collection supplies some information: his boyhood lasted one sixth of his life, his beard grew after one twelfth more, he married after one seventh more, and his son was born five years later. The son lived one half as long as the father, and the father died four years after the son. How long did Diophantus live?

§2 CONTINUOUS TRANSFORMATIONS IN THE PLANE

The topology in this book rests on just two fundamental concepts, both introduced informally in §1: continuity (or continuous transformation) and complex. These two notions, one analytical (pertaining to analysis or calculus) and one combinatorial, represent a genuine division of topology into two subjects: point set topology and combinatorial topology, each of which, although undoubtedly possessing close ties with the other, has enjoyed its own development. Point set topology is devoted to a close study of continuity. This study developed from the movement in the nineteenth century to place the calculus finally on a rigorous foundation. Previous generations of mathematicians going back to Newton and Leibnitz themselves had been too busy developing consequences of differentiation and integration, especially in connection with scientific applications, to pay much attention to these foundations. The result of this neglect was that there was not even agreement on the notion of function, much less on limits or continuity. In the nineteenth century several mathematicians resolved to do something about this situation. The leaders of this movement were Abel (1802–1829), Bolzano (1781–1848), Cauchy (1789–1857), and Weierstrass (1815–1897). The result of their activity was the precise formulation of the notions of function, limit, and derivative that are used today. The influence of set theory, the invention of Cantor (1845–1918) in the second half of the century, led to an abstract theory of continuity: point set topology. Today point set topology is an extensive independent field with applications to many parts of mathematics, particularly in analysis.

Combinatorial topology, on the other hand, developed at first as a branch of geometry. The work of Euler (§1) and a number of nineteenth-century geometers on polyhedra is part of this development. However, the foundations of the subject were laid by Poincaré (1854–1912) in a series of papers published around the turn of the century. Poincaré was motivated by problems in analysis, particularly in the qualitative theory of differential equations. We will follow a quasi-historical approach by presenting some of this theory in Chapter Two. Other names associated with the early development of combinatorial topology, whose ideas will be presented here, are Brouwer (1881–1967), Veblen (1880–1960), Alexander (1888–1971), and Lefschetz (1884–1972). The subsequent growth of combinatorial topology has been as extensive as point set topology with applications particularly in geometry and analysis. In addition, combinatorial topology in recent years has developed a strong algebraic flavor, which has led to the founding of new branches of algebra.

In the remainder of this chapter we present the concepts from point set topology, particularly the definition of continuous transformation, that are required in the rest of the book. At first we restrict ourselves to the plane. Ideas once established there will be easily generalized later. In the plane it makes sense to use Cartesian coordinates, so that every point P is associated with a pair of numbers $P = (x, y)$. Given two points P and $Q = (z, w)$, their sum is defined

$$P + Q = (x + z, y + w)$$

and the product of a point P and a real number t is defined by

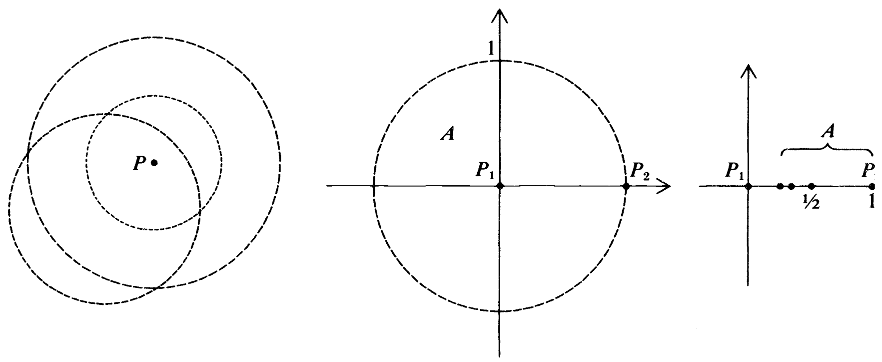
$$tP = (tx, ty)$$

These are the usual vector operations of addition and scalar multiplication. In addition, the **norm** of the point P is defined by

$$\|P\| = (x^2 + y^2)^{1/2}$$

This is the Euclidean distance from P to the origin. Using the norm we can express the distance between two points P and Q by $\|P - Q\|$.

Continuous transformations were characterized in §1 as transformations that do not involve any ripping or tearing. To put this in a positive way, continuous transformations must preserve the "nearness" of points; that is,



(a) (b) (c)

Figure 2.1 (a) A point P with some neighborhoods. (b) P_1 is in A and near A; P_2 is near A but not in A. (c) P_1 is near A; P_2 is near A and in A.

if a point is near a certain set, then the transformed point must be near the transformed set. To make this precise, all that must be done is to define exactly the notion of nearness. In following this approach we are using the ideas of the great analyst F. Riesz (1880–1956).

Definition

Let P be a point in the plane. A **neighborhood** *of P is any circular disk (without the boundary circle) that contains P (see Figure 2.1a). Let A be a subset of the plane. The point P is called* **near** *the set A if every neighborhood of P contains a point of A. If P is near A, we write* $P \leftarrow A$.

For example, let A be the open unit disk, that is, the set of points P such that $\|P\| < 1$. Then the points near A include all the points of A plus the points on the boundary circle $\|P\| = 1$ (see Figure 2.1b). On the other hand, if A is the set of points on the x-axis with x coordinates $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots$, then the only point near A, other than the points already in A, is the origin (see Figure 2.1c).

Exercises

1. Find the points near each of the following sets:

(a) $A =$ the circle $\|P\| = 1$