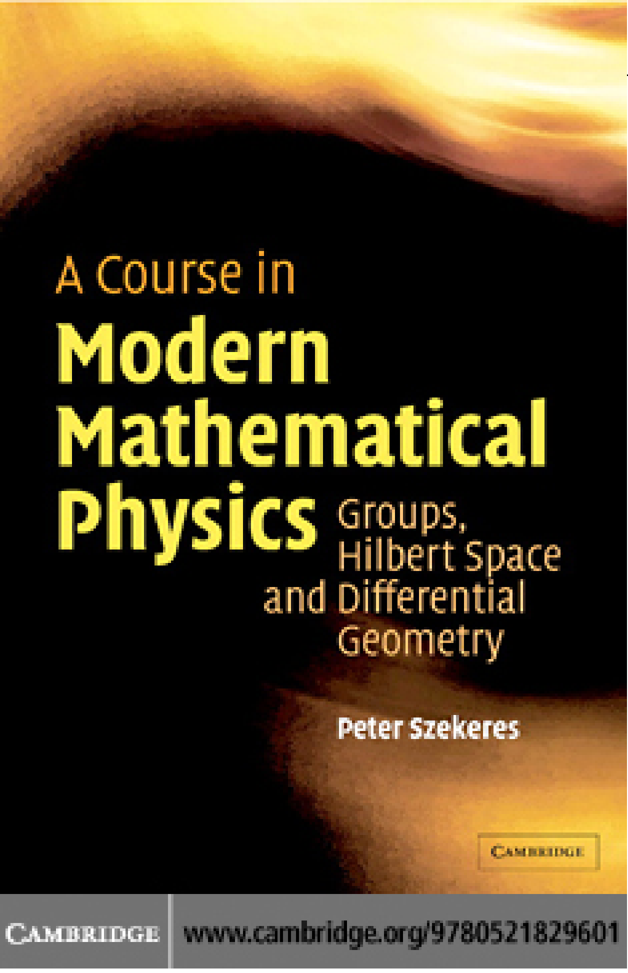
A Course in
Modern
Mathematical
Physics
Groups,
Hilbert Space
and Differential
Geometry
Peter Szekeres
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This book provides an introduction to the major mathematical structures used in physics today. It covers the concepts and techniques needed for topics such as group theory, Lie algebras, topology, Hilbert spaces and differential geometry. Important theories of physics such as classical and quantum mechanics, thermodynamics, and special and general relativity are also developed in detail, and presented in the appropriate mathematical language.

The book is suitable for advanced undergraduate and beginning graduate students in mathematical and theoretical physics. It includes numerous exercises and worked examples to test the reader's understanding of the various concepts, as well as extending the themes covered in the main text. The only prerequisites are elementary calculus and linear algebra. No prior knowledge of group theory, abstract vector spaces or topology is required.

PETER SZEKERES received his Ph.D. from King's College London in 1964, in the area of general relativity. He subsequently held research and teaching positions at Cornell University, King's College and the University of Adelaide, where he stayed from 1971 till his recent retirement. Currently he is a visiting research fellow at that institution. He is well known internationally for his research in general relativity and cosmology, and has an excellent reputation for his teaching and lecturing.

A Course in Modern Mathematical Physics

Groups, Hilbert Space and Differential Geometry

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Preface

After some twenty years of teaching different topics in the Department of Mathematical Physics at the University of Adelaide I conceived the rather foolhardy project of putting all my undergraduate notes together in one single volume under the title *Mathematical Physics*. This undertaking turned out to be considerably more ambitious than I had originally expected, and it was not until my recent retirement that I found the time to complete it.

Over the years I have sometimes found myself in the midst of a vigorous and at times quite acrimonious debate on the difference between theoretical and mathematical physics. This book is symptomatic of the difference. I believe that mathematical physicists put the mathematics first, while for theoretical physicists it is the physics which is uppermost. The latter seek out those areas of mathematics for the use they may be put to, while the former have a more unified view of the two disciplines. I don't want to say one is better than the other – it is simply a different outlook. In the big scheme of things both have their place but, as this book no doubt demonstrates, my personal preference is to view mathematical physics as a branch of mathematics.

The classical texts on mathematical physics which I was originally brought up on, such as Morse and Feshbach [7], Courant and Hilbert [1], and Jeffreys and Jeffreys [6] are essentially books on differential equations and linear algebra. The flavour of the present book is quite different. It follows much more the lines of Choquet-Bruhat, de Witt-Morette and Dillard-Bleick [14] and Geroch [3], in which mathematical structures rather than mathematical analysis is the main thrust. Of these two books, the former is possibly a little daunting as an introductory undergraduate text, while Geroch's book, written in the author's inimitably delightful lecturing style, has occasional tendencies to overabstraction. I resolved therefore to write a book which covers the material of these texts, assumes no more mathematical knowledge than elementary calculus and linear algebra, and demonstrates clearly how theories of modern physics fit into various mathematical structures. How well I have succeeded must be left to the reader to judge.

At times I have been caught by surprise at the natural development of ideas in this book. For example, how is it that quantum mechanics appears before classical mechanics? The reason is certainly not on historical grounds. In the natural organization of mathematical ideas, algebraic structures appear before geometrical or topological structures, and linear structures are evidently simpler than non-linear. From the point of view of mathematical simplicity quantum mechanics, being a purely linear theory in a quasi-algebraic space (Hilbert space), is more elementary than classical mechanics, which can be expressed in terms of non-linear dynamical systems in differential geometry. Yet, there is something of a paradox here, for as Niels Bohr remarked: ‘Anyone who is not shocked by quantum mechanics does not understand it’. Quantum mechanics is not a difficult theory to express mathematically, but it is almost impossible to make epistomological sense of it. I will not even attempt to answer these sorts of questions, and the reader must look elsewhere for a discussion of quantum measurement theory [5].

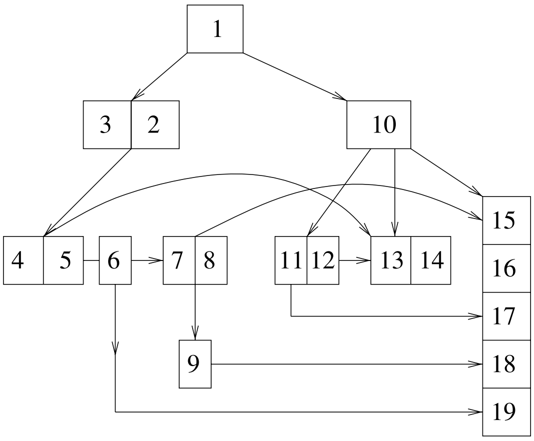
Every book has its limitations. At some point the author must call it a day, and the omissions in this book may prove a disappointment to some readers. Some of them are a disappointment to me. Those wanting to go further might explore the theory of fibre bundles and gauge theories [2, 8, 13], as the stage is perfectly set for this subject by the end of the book. To many, the biggest omission may be the lack of any discussion of quantum field theory. This, however, is an area that seems to have an entirely different flavour to the rest of physics as its mathematics is difficult if nigh on impossible to make rigorous. Even quantum mechanics has a ‘classical’ flavour by comparison. It is such a huge subject that I felt daunted to even begin it. The reader can only be directed to a number of suitable books to introduce them to this field [10–14].

Structure of the book

This book is essentially in two parts, modern algebra and geometry (including topology). The early chapters begin with set theory, group theory and vector spaces, then move to more advanced topics such as Lie algebras, tensors and exterior algebra. Occasionally ideas from group representation theory are discussed. If calculus appears in these chapters it is of an elementary kind. At the end of this algebraic part of the book, there is included a chapter on special relativity (Chapter 9), as it seems a nice example of much of the algebra that has gone before while introducing some notions from topology and calculus to be developed in the remaining chapters. I have treated it as a kind of crossroads: Minkowski space acts as a link between algebraic and geometric structures, while at the same time it is the first place where physics and mathematics are seen to interact in a significant way.

In the second part of the book, we discuss structures that are essentially geometrical in character, but generally have an algebraic component as well. Beginning with topology (Chapter 10), structures are created that combine both algebra and the concept of continuity. The first of these is Hilbert space (Chapter 13), which is followed by a chapter on quantum mechanics. Chapters on measure theory (Chapter 11) and distribution theory (Chapter 12) precede these two. The final chapters (15–19) deal with differential geometry and examples of physical theories using manifold theory as their setting – thermodynamics, classical mechanics, general relativity and cosmology. A flow diagram showing roughly how the chapters interlink is given below.

Exercises and problems are interspersed throughout the text. The exercises are not designed to be difficult – their aim is either to test the reader’s understanding of a concept just defined or to complete a proof needing one or two more steps. The problems at ends of sections are more challenging. Frequently they are in many parts, taking up a thread



of thought and running with it. This way most closely resembles true research, and is my preferred way of presenting problems rather than the short one-liners often found in text books. Throughout the book, newly defined concepts are written in bold type. If a concept is written in italics, it has been introduced in name only and has yet to be defined properly.

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To my mother, Esther

1 Sets and structures

The object of mathematical physics is to describe the physical world in purely mathematical terms. Although it had its origins in the science of ancient Greece, with the work of Archimedes, Euclid and Aristotle, it was not until the discoveries of Galileo and Newton that mathematical physics as we know it today had its true beginnings. Newton's discovery of the calculus and its application to physics was undoubtedly the defining moment. This was built upon by generations of brilliant mathematicians such as Euler, Lagrange, Hamilton and Gauss, who essentially formulated physical law in terms of differential equations. With the advent of new and unintuitive theories such as relativity and quantum mechanics in the twentieth century, the reliance on mathematics moved to increasingly recondite areas such as abstract algebra, topology, functional analysis and differential geometry. Even classical areas such as the mechanics of Lagrange and Hamilton, as well as classical thermodynamics, can be lifted almost directly into the language of modern differential geometry. Today, the emphasis is often more structural than analytical, and it is commonly believed that finding the right mathematical structure is the most important aspect of any physical theory. Analysis, or the consequences of theories, still has a part to play in mathematical physics – indeed, most research is of this nature – but it is possibly less fundamental in the total overview of the subject.

When we consider the significant achievements of mathematical physics, one cannot help but wonder why the workings of the universe are expressable at all by rigid mathematical 'laws'. Furthermore, how is it that purely human constructs, in the form of deep and subtle mathematical structures refined over centuries of thought, have any relevance at all? The nineteenth century view of a clockwork universe regulated deterministically by differential equations seems now to have been banished for ever, both through the fundamental appearance of probabilities in quantum mechanics and the indeterminism associated with chaotic systems. These two aspects of physical law, the deterministic and indeterministic, seem to interplay in some astonishing ways, the impact of which has yet to be fully appreciated. It is this interplay, however, that almost certainly gives our world its richness and variety. Some of these questions and challenges may be fundamentally unanswerable, but the fact remains that mathematics seems to be the correct path to understanding the physical world.

The aim of this book is to present the basic mathematical structures used in our subject, and to express some of the most important theories of physics in their appropriate mathematical setting. It is a book designed chiefly for students of physics who have the need for a more rigorous mathematical education. A basic knowledge of calculus and linear algebra, including matrix theory, is assumed throughout, but little else. While different students will of course come to this book with different levels of mathematical sophistication, the reader should be able to determine exactly what they can skip and where they must take pause. Mathematicians, for example, may be interested only in the later chapters, where various theories of physics are expressed in mathematical terms. These theories will not, however, be developed at great length, and their consequences will only be dealt with by way of a few examples.

The most fundamental notion in mathematics is that of a *set*, or ‘collection of objects’. The subject of this chapter is *set theory* – the branch of mathematics devoted to the study of sets as abstract objects in their own right. It turns out that every mathematical structure consists of a collection of sets together with some *defining relations*. Furthermore, as we shall see in Section 1.3, such relations are themselves defined in terms of sets. It is thus a commonly adopted viewpoint that all of mathematics reduces essentially to statements in set theory, and this is the motivation for starting with a chapter on such a basic topic.

The idea of sets as collections of objects has a non-rigorous, or ‘naive’ quality, although it is the form in which most students are introduced to the subject [1–4]. Early in the twentieth century, it was discovered by Bertrand Russell that there are inherent self-contradictions and paradoxes in overly simple versions of set theory. Although of concern to logicians and those mathematicians demanding a totally rigorous basis to their subject, these paradoxes usually involve inordinately large self-referential sets – not the sort of constructs likely to occur in physical contexts. Thus, while special models of set theory have been designed to avoid contradictions, they generally have somewhat artificial attributes and naive set theory should suffice for our purposes. The reader’s attention should be drawn, however, to the remarks at the end of Section 1.5 concerning the possible relevance of fundamental problems of set theory to physics. These problems, while not of overwhelming concern, may at least provide some food for thought.

While a basic familiarity with set theory will be assumed throughout this book, it nevertheless seems worthwhile to go over the fundamentals, if only for the sake of completeness and to establish a few conventions. Many physicists do not have a good grounding in set theory, and should find this chapter a useful exercise in developing the kind of rigorous thinking needed for mathematical physics. For mathematicians this is all bread and butter, and if you feel the material of this chapter is well-worn ground, please feel free to pass on quickly.

1.1 Sets and logic

There are essentially two ways in which we can think of a **set** S. Firstly, it can be regarded as a collection of mathematical objects a, b, . . . , called **constants**, written

$$S = \{a,\ b, \dots\}.$$

The constants $a,\ b, \dots$ may themselves be sets and, indeed, some formulations of set theory *require* them to be sets. Physicists in general prefer to avoid this formal nicety, and find it much more natural to allow for ‘atomic’ objects, as it is hard to think of quantities such as *temperature* or *velocity* as being ‘sets’. However, to think of sets as consisting of lists of objects is only suitable for finite or at most countably infinite sets. If we try putting the real numbers into a list we encounter the Cantor diagonalization problem – see Theorems 1.4 and 1.5 of Section 1.5.

The second approach to set theory is much more general in character. Let $P(x)$ be a *logical proposition* involving a **variable** x. Any such proposition symbolically defines a set

$$S = \{x \mid P(x)\},$$

which can be thought of as symbolically representing the collection of all x for which the proposition $P(x)$ is true. We will not attempt a full definition of the concept of logical proposition here – this is the business of formal logic and is only of peripheral interest to theoretical physicists – but some comments are in order. Essentially, logical propositions are statements made up from an alphabet of symbols, some of which are termed **constants** and some of which are called **variables**, together with logical connectives such as **not**, **and**, **or** and **implies**, to be manipulated according to rules of standard logic. Instead of 'P implies Q' we frequently use the words '**if** P **then** Q' or the symbolic representation $P \Rightarrow Q$. The statement 'P **if and only if** Q', or 'P **iff** Q', symbolically written $P \Leftrightarrow Q$, is a shorthand for

$$(P \Rightarrow Q) \text{ and } (Q \Rightarrow P),$$

and signifies logical equivalence of the propositions P and Q. The two **quantifiers** $\forall$ and $\exists$, said **for all** and **there exists**, respectively, make their appearance in the following way: if $P(x)$ is a proposition involving a variable x, then

$$\forall x(P(x)) \quad \text{and} \quad \exists x(P(x))$$

are propositions.

Mathematical theories such as *set theory*, *group theory*, etc. traditionally involve the introduction of some new symbols with which to generate further logical propositions. The theory must be complemented by a collection of logical propositions called **axioms** for the theory – statements that are taken to be automatically **true** in the theory. All other true statements should in principle follow by the rules of logic.

Set theory involves the introduction of the new phrase **is a set** and new symbols $\{\dots \mid \dots\}$ and $\in$ defined by:

(Set1) If S is any constant or variable then 'S is a set' is a logical proposition.
(Set2) If $P(x)$ is a logical proposition involving a variable x then $\{x \mid P(x)\}$ is a set.
(Set3) If S is a set and a is any constant or variable then $a \in S$ is a logical proposition, for which we say a **belongs to** S or a **is a member of** S, or simply a **is in** S. The negative of this proposition is denoted $a \notin S$ – said a **is not in** S.

These statements say nothing about whether the various propositions are true or false – they merely assert what are 'grammatically correct' propositions in set theory. They merely tell us how the new symbols and phrases are to be used in a grammatically correct fashion. The main axiom of set theory is: if $P(x)$ is any logical proposition depending on a variable x, then for any constant or variable a

$$a \in \{x \mid P(x)\} \Leftrightarrow P(a).$$

Every mathematical theory uses the equality symbol $=$ to express the identity of mathematical objects in the theory. In some cases the concept of mathematical identity needs a separate definition. For example **equality of sets** $A = B$ is defined through the *axiom of extensionality*:

Two sets A and B are equal if and only if they contain the same members. Expressed symbolically,

$$A = B \Leftrightarrow \forall a (a \in A \Leftrightarrow a \in B).$$

A **finite set** $A = \{a_1, a_2, \dots, a_n\}$ is equivalent to

$$A = \{x \mid (x = a_1) \text{ or } (x = a_2) \text{ or } \dots \text{ or } (x = a_n)\}.$$

A set consisting of just one element a is called a **singleton** and should be written as $\{a\}$ to distinguish it from the element a which belongs to it: $\{a\} = \{x \mid x = a\}$.

As remarked above, sets can be members of other sets. A set whose elements are all sets themselves will often be called a **collection** or **family** of sets. Such collections are often denoted by script letters such as $\mathcal{A}$, $\mathcal{U}$, etc. Frequently a family of sets $\mathcal{U}$ has its members **indexed** by another set I, called the **indexing set**, and is written

$$\mathcal{U} = \{U_i \mid i \in I\}.$$

For a finite family we usually take the indexing set to be the first *n natural numbers*, $I = \{1, 2, \dots, n\}$. Strictly speaking, this set must also be given an axiomatic definition such as *Peano's axioms*. We refer the interested reader to texts such as [4] for a discussion of these matters.

Although the finer details of logic have been omitted here, essentially all concepts of set theory can be constructed from these basics. The implication is that all of mathematics can be built out of an alphabet for constants and variables, parentheses $(\dots)$, logical connectives and quantifiers together with the rules of propositional logic, and the symbols $\{\dots \mid \dots\}$ and $\in$. Since mathematical physics is an attempt to express physics in purely mathematical language, we have the somewhat astonishing implication that all of physics should also be reducible to these simple terms. Eugene Wigner has expressed wonderment at this idea in a famous paper entitled *The unreasonable effectiveness of mathematics in the natural sciences* [5].

The presentation of set theory given here should suffice for all practical purposes, but it is not without logical difficulties. The most famous is *Russell's paradox*: consider the set of all sets which are not members of themselves. According to the above rules this set can be written $R = \{A \mid A \notin A\}$. Is R a member of itself? This question does not appear to have an answer. For, if $R \in R$ then by definition $R \notin R$, which is a contradiction. On the other hand, if $R \notin R$ then it satisfies the criterion required for membership of R; that is, $R \in R$.

To avoid such vicious arguments, logicians have been forced to reformulate the axioms of set theory in a very careful way. The most frequently used system is the axiomatic scheme of *Zermelo and Fraenkel* – see, for example, [2] or the Appendix of [6]. We will adopt the 'naive' position and simply assume that the sets dealt with in this book do not exhibit the self-contradictions of Russell's monster.

1.2 Subsets, unions and intersections of sets

A set T is said to be a **subset** of S, or T is **contained in** S, if every member of T belongs to S. Symbolically, this is written $T \subseteq S$,

$$T \subseteq S \quad \text{iff} \quad a \in T \Rightarrow a \in S.$$

We may also say S is a **superset** of T and write $S \supset T$. Of particular importance is the **empty set** $\emptyset$, to which no object belongs,

$$\forall a\,(a \notin \emptyset).$$

The empty set is assumed to be a subset of any set whatsoever,

$$\forall S(\emptyset \subseteq S).$$

This is the default position, consistent with the fact that $a \in \emptyset \Rightarrow a \in S$, since there are no a such that $a \in \emptyset$ and the left-hand side of the implication is never true. We have here an example of the logical dictum that 'a false statement implies the truth of any statement'.

A common criterion for showing the equality of two sets, $T = S$, is to show that $T \subseteq S$ and $S \subseteq T$. The proof follows from the axiom of extensionality:

$$\begin{aligned} T = S &\iff (a \in T \Leftrightarrow a \in S) \\ &\iff (a \in T \Rightarrow a \in S) \text{ and } (a \in S \Rightarrow a \in T) \\ &\iff (T \subseteq S) \text{ and } (S \subseteq T). \end{aligned}$$

Exercise: Show that the empty set is unique; i.e., if $\emptyset'$ is an empty set then $\emptyset' = \emptyset$.

The collection of all subsets of a set S forms a set in its own right, called the **power set** of S, denoted 2^S.

Example 1.1 If S is a finite set consisting of n elements, then 2^S consists of one empty set $\emptyset$ having no elements, n singleton sets having just one member, $\binom{n}{2}$ sets having two elements, etc. Hence the total number of sets belonging to 2^S is, by the binomial theorem,

$$1 + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n} = (1+1)^n = 2^n.$$

This motivates the symbolic representation of the power set.

Unions and intersections

The **union** of two sets S and T, denoted $S \cup T$, is defined as

$$S \cup T = \{x \mid x \in S \text{ or } x \in T\}.$$

The **intersection** of two sets S and T, denoted $S \cap T$, is defined as

$$S \cap T = \{x \mid x \in S \text{ and } x \in T\}.$$

Two sets S and T are called **disjoint** if no element belongs simultaneously to both sets, $S \cap T = \emptyset$. The **difference** of two sets S and T is defined as

$$S - T = \{x \mid x \in S \text{ and } x \notin T\}.$$

Exercise: If S and T are disjoint, show that $S - T = S$.

The union of an arbitrary (possibly infinite) family of sets $\mathcal{A}$ is defined as the set of all elements x that belong to some member of the family,

$$\bigcup \mathcal{A} = \{x \mid \exists S \text{ such that } (S \in \mathcal{A}) \text{ and } (x \in S)\}.$$

Similarly we define the **intersection** of $\mathcal{S}$ to be the set of all elements that belong to *every* set of the collection,

$$\bigcap \mathcal{A} = \{x \mid x \in S \text{ for all } S \in \mathcal{A}\}.$$

When $\mathcal{A}$ consists of a family of sets S_i indexed by a set I, the union and intersection are frequently written

$$\bigcup_{i \in I} \{S_i\} \quad \text{and} \quad \bigcap_{i \in I} \{S_i\}.$$

Problems

Problem 1.1 Show the distributive laws

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C), \quad A \cup (B \cap C) = (A \cup B) \cap (A \cup C).$$

Problem 1.2 If $\mathcal{B} = \{B_i \mid i \in I\}$ is any family of sets, show that

$$A \cap \bigcup \mathcal{B} = \bigcup \{A \cap B_i \mid i \in I\}, \quad A \cup \bigcap \mathcal{B} = \bigcap \{A \cup B_i \mid i \in I\}.$$

Problem 1.3 Let B be any set. Show that $(A \cap B) \cup C = A \cap (B \cup C)$ if and only if $C \subseteq A$.

Problem 1.4 Show that

$$A - (B \cup C) = (A - B) \cap (A - C), \quad A - (B \cap C) = (A - B) \cup (A - C).$$

Problem 1.5 If $\mathcal{B} = \{B_i \mid i \in I\}$ is any family of sets, show that

$$A - \bigcup \mathcal{B} = \bigcup \{A - B_i \mid i \in I\}.$$

Problem 1.6 If E and F are any sets, prove the identities

$$2^E \cap 2^F = 2^{E \cap F}, \quad 2^E \cup 2^F \subseteq 2^{E \cup F}.$$

Problem 1.7 Show that if $\mathcal{C}$ is any family of sets then

$$\bigcap_{X \in \mathcal{C}} 2^X = 2^{\bigcap \mathcal{C}}, \quad \bigcup_{X \in \mathcal{C}} 2^X \subseteq 2^{\bigcup \mathcal{C}}.$$