Universitext

UTX

Mark de Longueville

A Course in Topological Combinatorics

Springer

Mark de Longueville, Universitext, *A Course in Topological Combinatorics*, 2013, DOI: 10.1007/978-1-4419-7910-0, © Springer Science+Business Media New York 2013

Universitext

Universitext is a series of textbooks that presents material from a wide variety of mathematical disciplines at master's level and beyond. The books, often well class-tested by their author, may have an informal, personal, even experimental approach to their subject matter. Some of the most successful and established books in the series have evolved through several editions, always following the evolution of teaching curricula, into very polished texts.

Thus as research topics trickle down into graduate-level teaching, first textbooks written for new, cutting-edge courses may make their way into *Universitext* .

For further volumes: http://www.springer.com/series/223

Mark de Longueville

A Course in Topological Combinatorics

Springer

Mark de Longueville
Hochschule für Technik und Wirtschaft Berlin, University of Applied Sciences, Berlin, Germany

ISSN 0172-5939 e-ISSN 2191-6675

ISBN 978-1-4419-7909-4 e-ISBN 978-1-4419-7910-0

Springer New York Heidelberg Dordrecht London

Library of Congress Control Number: 2012937863

Printed on acid-free paper

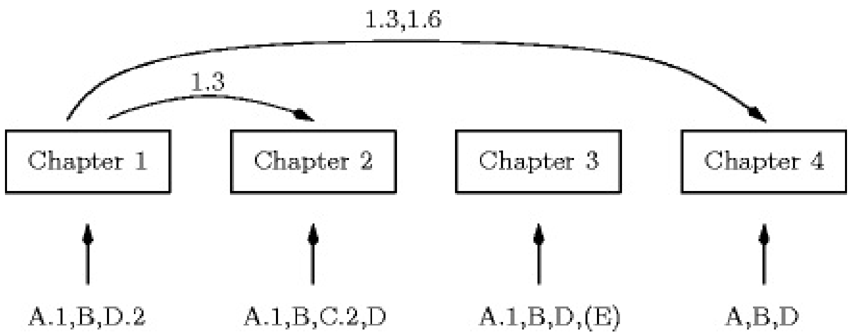
Springer is part of Springer Science+Business Media (www.springer.com)

For ~~L., with whom I fell, first from the bicycle and then in love~~

Preface

Topological combinatorics is a very young and exciting field of research in mathematics. It is mostly concerned with the application of the many powerful tools of algebraic topology to combinatorial problems. One of its early landmarks was Lovász's proof of the Kneser conjecture published in 1978 The combination of the two mathematical fields—topology and combinatorics—has led to many surprising and elegant proofs and results.

In this textbook I present some of the most beautiful and accessible results from topological combinatorics. It grew out of several courses that I have taught at Freie Universität Berlin, and is based on my personal taste and what I believe is suitable for the classroom. In particular, it aims for clear and vivid presentation rather than encyclopedic completeness.



The text is designed for an advanced undergraduate level. Primarily it serves as a basis for a course, but is written in such a way that it just as well may be read by students independently. The textbook is essentially self-contained. Only some basic mathematical experience and knowledge—in particular some linear algebra—is required. An extensive appendix allows the instructor to design courses for students with very different prerequisites. Some of those designs will be sketched later o

The textbook has four main chapters and several appendices. Each chapter ends with an accompanying and complementing set of exercises. The main chapters are mostly independent of eac other and thus allow considerable flexibility for an individual course design. The dependencies are roughly as follows.

Suggested Course Outlines

For students with previous knowledge of graph theory and the basics of algebraic topology including simplicial homology theory. Use Chaps. 1 – 4 . Whenever concepts and results on partially ordered se and their topology from Appendix C or on group actions from Appendix D are missing, they should b included. Oliver's Theorem 3.17, which is proven in Appendix E, can easily be used as a black box. I the students are experienced with homology and if time permits, I recommend studying Appendix E after Chap. 3 . *For students with previous knowledge of the basics of algebraic topology including simplicial homology theory only*. Proceed as in the last case and provide the basics of graph theory from Appendix A along the way. *For students with previous knowledge of graph theory only*. I recommend that the instructor introduces some basic topology with Sects. B.1 and B.3, and then presents Chap. 1 , skipping the homological proofs. Before Sect. 1.6 I recommend giving a topology crash course with Sects. B.4–B.9. Proceed with Chaps. 2 – 4 and add concepts and results from Appendices C and D as needed. Apply Theorem 3.17 as a black box and use Appendix E as a motivation to convince students to study algebraic topology. *For motivated students with neither* *graph theory nor algebraic topology knowledge.* Proceed as in the last case and provide the basics from graph theory from Appendix A along the way.

Acknowledgments

First of all, I would like to thank all the authors of research papers and textbooks—several of them I know personally—on which this book is based. I am thankful to Martin Aigner for helpful advice anc for supporting the initial idea of the project, and to Günter M. Ziegler for providing excellent workin conditions in his research group at Technische Universität Berlin. I am indebted to all the students who took part in my courses on the subject, and to all of the colleagues who helped me with discussions, suggestions, and proofreading. In particular, I want to thank Anna Gundert, Nicolina Hauke, Daria Schymura, Felix Breuer, Aaron Dall, Anton Dochtermann, Frederik von Heymann, Fran Lutz, Benjamin Matschke, Marc Pfetsch, and Carsten Schultz. I also want to thank David Kramer fro Springer New York for his very helpful copy editing, and finally, Hans Koelsch and Kaitlin Leach from Springer New York for their competent support of this project.

Mark de Longuevil

List of Symbols and Typical Notation

$[n] = \{1, 2, \ldots, n\}$ the set of natural numbers from 1 to n

$|S|$ the number of elements of a set S

$\lfloor x \rfloor$ the largest integer less than or equal to x

$k \mid n$ notation for "k divides n"

$\subseteq$ the subset relation

$\subset$ the proper subset relation

$S = S_1 \dot\cup \cdots \dot\cup S_n$ a partition of the set S, i.e., $S = S_1 \cup \cdots \cup S_n$ and $S_i \cap S_j = \emptyset$ for all $i \neq j$

$\binom{n}{k} = \frac{n!}{k!(n-k)!}$ the number of k-element subsets of an n-element set

$\binom{X}{k} = \{S \subseteq X : |S| = k\}$ the set of k-element subsets of a set X

$X + Y = X \times \{0\} \cup Y \times \{1\}$ the sum of sets X and Y

K, L abstract simplicial complexes

Δ, Γ geometric simplicial complexes

$\tau \leq \sigma$ notation for "the simplex τ is a face of the simplex σ"

$\sigma^n = conv(\{e_1, \ldots, e_{n+1}\})$ the standard geometric n-simplex

Δ^n the geometric simplicial complex given by σ^n and all its faces

$K(\Delta)$ the abstract simplicial complex associated with the geometric complex Δ (cf. page 177)

$|\Delta|$ the polyhedron of the geometric simplicial complex Δ

$|K|$ a geometric realization of the abstract complex K or its polyhedron

$\mathcal{P}(X)$ the power set of X, i.e., $\mathcal{P}(X) = \{A : A \subseteq X\}$

2^X will be identified with the power set of X

$2^{[n]}$ will be identified with the power set of $[n]$

2^σ will be identified with the power set of σ, in this notation refers to the abstract simplicial complex given by the simplex σ and all its faces

$\|\cdot\| = \|\cdot\|_2$ the Euclidean norm

$\|\cdot\|_\infty$ the maximum norm

$\mathbb{B}^n = \{x \in \mathbb{R}^n : \|x\| \leq 1\}$ the n-dimensional unit ball

$\mathbb{S}^{n-1} = \{x \in \mathbb{R}^n : \|x\| = 1\}$ the $(n-1)$-dimensional unit sphere

$Q^{n+1} = conv(\{\pm e_1, \ldots, \pm e_{n+1}\})$ the $(n+1)$-dimensional cross polytope

Γ^n the n-dimensional geometric simplicial complex associated with the boundary of the cross polytope Q^{n+1}

Contents

Mark de Longueville, Universitext, *A Course in Topological Combinatorics*, 2013, DOI: 10.1007/978-1-4419-7910-0_1, © Springe
Science+Business Media New York 2013

1. Fair-Division Problems

Mark de Longueville[1]

(1) Hochschule für Technik und Wirtschaft Berlin, University of Applied Sciences, Berlin, German

Abstract
Almost every day, we encounter fair-division problems: in the guise of dividing a piece of cake, slicing a ham sandwich, or by dividing our time with respect to the needs and expectations of family, friends, work, etc.

Almost every day, we encounter fair-division problems: in the guise of dividing a piece of cake, slicing a ham sandwich, or by dividing our time with respect to the needs and expectations of family, friends, work, etc.

The mathematics of such fair-division problems will serve us as a first representative example fo the interplay between combinatorics and topology.

In this chapter we will consider two important concepts: *envy-free fair division* and *consensus division*. These concepts lead to different topological tools that we may apply. On the one hand, there is Brouwer's fixed-point theorem, and on the other hand, there is the theorem of Borsuk and Ulam. These topological results surprisingly turn out to have combinatorial analogues: the lemmas of Sperner and Tucker. Very similar in nature, they guarantee a simplex with a certain labeling in a labeled simplicial complex.

The chapter is organized in such a way that we will discuss in turn a topological result, its combinatorial analogue, and the corresponding fair-division problem.

1.1 Brouwer's Fixed-Point Theorem and Sperner's Lemma

Brouwer's fixed-point theorem states that any continuous map from a ball of any dimension to itself has a fixed point. In two dimensions this can be illustrated as follows. Take two identical maps of Berlin or any other ball-shaped city. Now crumple one of the maps as you like and throw it on the other, flat, map as shown in Fig. 1.1. Then there exists a location in the city that on the crumpled map is exactly above the same place on the flat map.

Fig. 1.1 A city map twice

For the general formulation of Brouwer's theorem, recall that the *n*-dimensional Euclidean ball i given by all points of distance at most 1 from the origin in *n*-dimensional Euclidean space, i.e., $\mathbb{B}^n = \{x \in \mathbb{R}^n : \|x\| \leq 1\}$.

Theorem 1.1 (Brouwer).
Every continuous map $f : \mathbb{B}^n \to \mathbb{B}^n$ *from the n-dimensional ball* $\mathbb{B}^n$ *to itself has a fixed point, i.e., ther exists an* $x \in \mathbb{B}^n$ *such that f(x) = x.*

The first proof that we provide for this theorem relies on a beautiful combinatorial lemma that we wi discuss in the next section. There also exists a very short and simple proof using homology theory th we present on page 20.

Sperner's Lemma

Brouwer's fixed-point theorem is intimately related to a combinatorial lemma by Sperner that deals with labelings of triangulations of the simplex. Consider the standard *n*-simplex given as the convex hull of the standard basis vectors $\{e_1, \ldots, e_{n+1}\} \subseteq \mathbb{R}^{n+1}$, see Fig. 1.2:

$$\begin{aligned}\sigma^n &= \operatorname{conv}\left(\{e_1, \ldots, e_{n+1}\}\right) \\ &= \left\{t_1 e_1 + \cdots + t_{n+1} e_{n+1} : t_i \geq 0, \sum_{i=1}^{n+1} t_i = 1\right\} \\ &= \left\{(t_1, \ldots, t_{n+1}) : t_i \geq 0, \sum_{i=1}^{n+1} t_i = 1\right\}.\end{aligned}$$

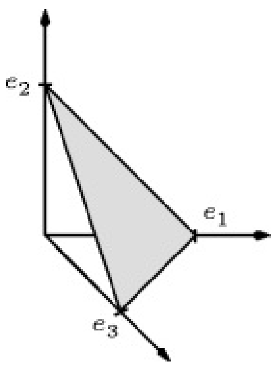


Fig. 1.2 The standard 2-simplex $\sigma^2 = \operatorname{conv}(\{e_1, e_2, e_3\})$

By Δ^n we denote the (geometric) simplicial complex given by σ^n and all its faces, i.e., $\Delta^n = \{\tau : \tau \leq \sigma^n\}$. Assume that *K* is a subdivision of Δ^n. We may think of *K* as being obtained from Δ^n by adding extra vertices. For precise definitions and more details on simplicial complexes we refer to Appendix B. For any *n*, denote the set {1, …, *n*} by [*n*]. In the definition of a Sperner labeling we will use labels from 1 to *n* + 1, i.e., labels from the set [*n* + 1].

Definition 1.2.
A *Sperner labeling* is a labeling $\lambda : \operatorname{vert}(K) \to [n+1]$ of the vertices of *K* satisfying

$$\lambda(v) \in \{i \in [n+1] : v_i \neq 0\}$$

for all $v \in \mathrm{vert}(K)$.

More intuitively, a Sperner labeling is the following. Consider the minimal face of Δ^n that contains v. Say it is given by the convex hull of $e_{i_1}, \dots, e_{i_k}$. Then v is allowed to get labels only from $\{i_1, \dots, i_k\}$. In particular, the vertices e_i obtain label i, while a vertex along the edge spanned by e_i and e_j obtains the label i or j, and so on. For an illustration see Fig. 1.3.

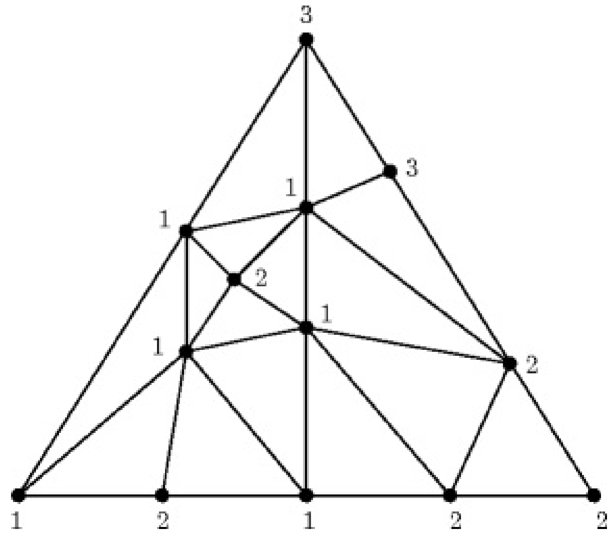


Fig. 1.3 A Sperner-labeled triangulation of a 2-simplex

Call an n-simplex of *Kfully labeled* (with respect to λ) if its $n + 1$ vertices obtain distinct labels, i.e., if all possible labels from the set $[n + 1]$ appear.

Lemma 1.3 (Sperner [Spe28]).
Let $\lambda : \mathrm{vert}\,(K) \to [n + 1]$ *be a Sperner labeling of a triangulation K of the n-dimensional simplex. Then there exists a fully labeled n-simplex in K. More precisely, the number of fully labeled n-simplices is odd.*

We will now present two inductive proofs of this amazing lemma. The first is a combinatorial construction that constructs one of the desired simplices, while the other is algebraic and uses the concept of a chain complex of a simplicial complex. The inductive proofs reveal a typical phenomenon: while we are mainly interested in the existence of a fully labeled simplex, the induction works only for the stronger statement that there is an odd number of fully labeled simplices.

A third proof, given on page 21 of this section, proves the Sperner lemma as an application of Brouwer's fixed-point theorem.

Proof (combinatorial).
The lemma is clearly valid for $n = 1$. Now let $n \geq 2$ and consider the $(n - 1)$-dimensional face τ of Δ^n given by the convex hull of $e_1, \dots, e_n$. Note that K restricted to τ is Sperner labeled with label set $[n]$. We construct a graph as follows. Let the vertex set be all n-simplices of K plus one extra vertex o. The extra vertex o is connected by an edge to all n-simplices that have an $(n - 1)$-simplex as a face that is labeled with all labels of $[n]$ and lies within τ. Two n-simplices are connected by an edge if and only if they share an $(n - 1)$-dimensional face labeled with all of $[n]$. See Fig. 1.4 for an example of the resulting graph.

By the induction hypothesis, the vertex o has odd degree, since there is an odd number of fully labeled ~~simplices in the labeling restricted to τ. All the other vertices have degree zero, one, or two. To see this~~ consider the set of labels an n-simplex obtains: either it does not contain $[n]$, it is $[n+1]$, or it is $[n]$. In the first case, the simplex has degree zero; in the second, it has degree one; and in the last case, it has degree two, since exactly two faces obtain all of $[n]$ as label set; compare Fig. 1.5. Hence the vertices of degree one other than o (which may have degree one) correspond to the fully labeled simplices. Now the number of vertices of odd degree in a graph is even. (This is easy to prove; cf. Corollary A.2 in Appendix A.) Since the degree of o is odd, there remains an odd number of fully labeled simplices.

Proof (algebraic).

We proceed by induction. The case $n = 1$ is an easy exercise. Now assume $n \geq 2$. The labeling λ induces a simplicial map from K to Δ^n defined on the vertices by $v \mapsto e_{\lambda(v)}$. Consider the induced map λ_* on the $\mathbb{Z}_2$-simplicial chain complex level

$$\lambda_* : C_*(K;\mathbb{Z}_2) \to C_*(\Delta^n;\mathbb{Z}_2).$$

Let o denote the element of $C_n(K;\mathbb{Z}_2)$ given by the sum of all n-simplices of K. Clearly, the Sperner lemma holds if $\lambda_n(o) = \sigma^n$, the generator (and only nontrivial element) of $C_n(\Delta^n;\mathbb{Z}_2)$. Now consider the following diagram, which is commutative by the fact that λ_* is a chain map:

$$\begin{array}{ccc} C_n(K;\mathbb{Z}_2) & \xrightarrow{\lambda_n} & C_n(\Delta^n;\mathbb{Z}_2) \\ \partial_n & & \partial_n \\ C_{n-1}(K;\mathbb{Z}_2) & \xrightarrow{\lambda_{n-1}} & C_{n-1}(\Delta^n;\mathbb{Z}_2) \end{array}$$

Hence, it suffices to show that $\lambda_{n-1}\partial_n(o) \neq 0$. In order to compute $\lambda_{n-1}\partial_n(o)$, let $\tau_1, \ldots, \tau_{n+1}$ denote the $(n-1)$-dimensional faces of Δ^n. Define $c_i \in C_{n-1}(K;\mathbb{Z}_2)$ to be the sum of all $(n-1)$-dimensional faces of K that lie in τ_i. Then $\partial_n(o) = \sum_{i=1}^{n+1} c_i$, and by the induction hypothesis, $\lambda_{n-1}(c_i) = \tau_i$, and hence $\lambda_{n-1}\partial_n(o) = \sum_{i=1}^{n+1} \tau_i \neq 0$.

Brouwer's Theorem via Sperner's Lemma

Finally, we can give an elementary proof of Brouwer's fixed-point theorem relying on Sperner's lemma.

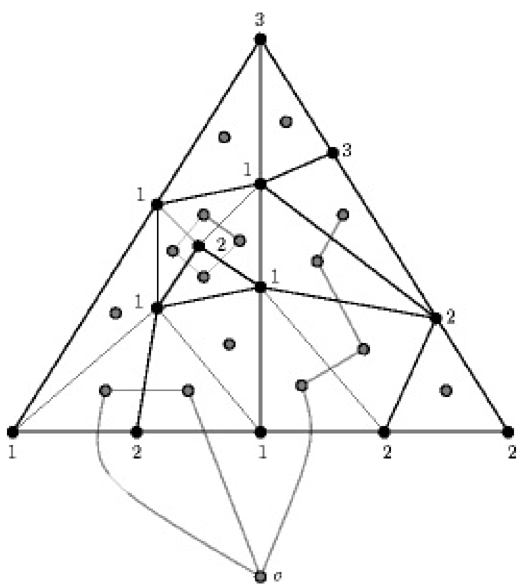


Fig. 1.4 The graph associated to a Sperner labeling

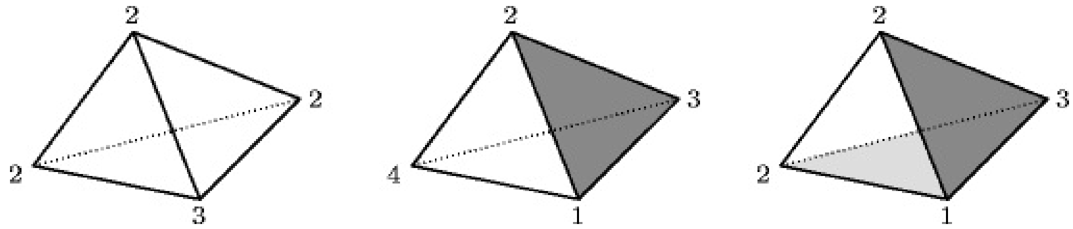


Fig. 1.5 An example of labeled n-simplices of degrees 0, 1, and 2 in the case $n = 3$

Proof (of Brouwer's fixed-point theorem).
Since $\mathbb{B}^n$ and the standard n-simplex are homeomorphic, we may consider a continuous map $f : |\Delta^n| \to |\Delta^n|$, where Δ^n is the (geometric) simplicial complex given by the standard n-simplex and all its faces. Consider the kth barycentric subdivisions $\mathrm{sd}^k \Delta^n$, $k \geq 1$. If, for some k, one of the vertices of $\mathrm{sd}^k \Delta^n$ happens to be a fixed point, we are done. Otherwise, we construct a sequence $(\sigma_k)_{k \geq 1}$ of simplices of decreasing size such that any *accumulation point* of this sequence will be a fixed point of f. By an accumulation point we mean a point $x \in |\Delta^n|$ such that each ε-ball about x contains infinitely many of the σ_k, $k \geq 1$. In order to do this, we endow the kth barycentric subdivision $\mathrm{sd}^k \Delta^n$ with a Sperner labeling as follows. For $v \in \mathrm{vert}(\mathrm{sd}^k \Delta^n)$ let $\lambda(v)$ be the smallest i such that the ith coordinate of $f(v) - v$ is negative, i.e.,

$$\lambda(v) = \min\{i : f(v)_i - v_i < 0\}.$$

Such an i exists, since the sum over all coordinates of $f(v) - v$ is zero and v is not a fixed point. This labeling is indeed a Sperner labeling, since for $v_i = 0$, we certainly have $f(v)_i - v_i \geq 0$. Hence, by Sperner's lemma, there exists a fully labeled simplex σ_k.

Now let x be an accumulation point of the sequence (σ_k) of simplices. For the existence of such an x we refer to Corollary B.48 and Exercise 16 on page 195. Hence, for each i and any $\varepsilon > 0$, there exist $k \geq 1$ and a vertex $v \in \mathrm{vert}(\sigma_k)$ such that $|x - v| < \varepsilon$ and $f(v)_i - v_i < 0$. By continuity, we obtain the inequality $f(x)_i - x_i \leq 0$ for all i. But since the sum $\sum_{i=1}^{n+1}(f(x)_i - x_i)$ is zero, this is possible only if $f(x) = x$.

Brouwer's Theorem via Homology Theory

As previously announced, we provide a proof of Brouwer's theorem using only the basics of homology theory typically taught in a first course on algebraic topology. More details on the necessary background can be found in Appendix B.

Proof (using homology theory).
Assume that $f : \mathbb{B}^n \to \mathbb{B}^n$ is a continuous map without a fixed point. For each x, consider the ray from $f(x)$ in the direction of x. This ray hits the boundary sphere $\mathbb{S}^{n-1} = \{x \in \mathbb{R}^n : \|x\| = 1\}$ of $\mathbb{B}^n$ in a point that we call $r(x)$; see Fig. 1.6. Then $r : \mathbb{B}^n \to \mathbb{S}^{n-1}$ is a continuous map that when restricted to the sphere, is the identity map, i.e., $r \circ i = \mathrm{id}_{\mathbb{S}^{n-1}}$, where $i : \mathbb{S}^{n-1} \hookrightarrow \mathbb{B}^n$ is the inclusion map. Such a map is called a retraction map. We obtain the following induced maps in homology:

$$\tilde{H}^{n-1}(\mathbb{S}^{n-1};\mathbb{Z}) \xrightarrow{i_*} \tilde{H}^{n-1}(\mathbb{B}^n;\mathbb{Z}) \xrightarrow{r_*} \tilde{H}^{n-1}(\mathbb{S}^{n-1};\mathbb{Z}).$$

Now, $r_* \circ i_* = (r \circ i)_* = (\mathrm{id}_{\mathbb{S}^{n-1}})_* = \mathrm{id}_{\tilde{H}^{n-1}(\mathbb{S}^{n-1};\mathbb{Z})}$ is the identity map of $\tilde{H}^{n-1}(\mathbb{S}^{n-1};\mathbb{Z})$, which is isomorphic to the integers $\mathbb{Z}$. But since $\tilde{H}^{n-1}(\mathbb{B}^n;\mathbb{Z})$ is trivial, we arrive at a contradiction.

Sperner's Lemma Derived from Brouwer's Theorem

As remarked in the introduction to this chapter, Sperner's lemma may be considered a combinatorial analogue of Brouwer's theorem. This is due to the fact that there is also a way to deduce Sperner's lemma from Brouwer's fixed-point theorem. We end this section by proving this.

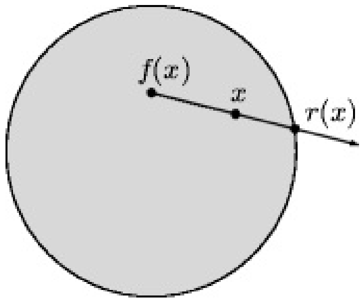


Fig. 1.6 The retraction map *r*

Proof (of Sperner's lemma with Brouwer's fixed-point theorem).
Let $\lambda : \mathrm{vert}(K) \to [n+1]$ be a Sperner labeling of a triangulation *K* of Δ^n. We construct a continuous map $f : |\Delta^n| \to |\Delta^n|$ as an affine linear extension of the simplicial map from *K* to Δ^n defined on the vertices of *K* by $v \mapsto e_{\lambda(v)+1(\mathrm{mod}\, n+1)}$.

Observe that there exists a fully labeled simplex if and only if *f* is surjective. To prove this, it suffices to show that some point in the interior of $|\Delta^n|$ is in the image of *f*. It is a good exercise to show that the map *f* is fixed-point-free on the boundary of $|\Delta^n|$. The existence of a fixed point yields the desired conclusion.

1.2 Envy-Free Fair Division

Sometimes it is hard to divide a piece of cake among several people, especially if the cake contains tasty ingredients such as nuts, raisins, and chocolate chips that may be distributed unevenly and if we take into account that preferences among several people are often quite different. This calls for a procedure to find a solution that is satisfying for everyone. In order to do this, we first have to specify more precisely what we mean by "satisfying for everyone." Ludwig Erhard, German chancellor in the 1960s, once said, "Compromise is the art of dividing a cake such that everyone is of the opinion he has received the largest piece." This seemingly paradoxical statement is pretty much what our definition of envy-free fair division is going to be!

A Fair-Division Model

Let's say we have *n* people among whom the cake is to be divided. Each person might have his or her own idea about which content of the cake is valuable: for one it's the nuts, for another it's the chocolate, and so on. Figure 1.7 shows a cake with colored chocolate beans indicating the different preferences. We model the piece of cake with an interval $I = [0, 1]$ (which might be thought of as a projection of the cake), and the predilections of the people by *continuous probability measures* μ_1,

$\dots,\mu_n$. Continuous means that the functions $t \mapsto \mu_i([0,t))$ are continuous in t. The continuity conditio implies that the measures evaluate to zero on single points, i.e., $\mu_i(\{t\}) = 0$ for all $t \in I$.

Fig. 1.7 A cake with different tasty ingredients

Let's assume that the cake is divided into n pieces (each measurable with respect to all μ_i), i.e., $I = A_1 \cup \cdots \cup A_n$, where $A_i \cap A_j$ is a finite set of points for each $i \neq j$, and person i is to receive the piece $A_{\pi(i)}$ for some permutation π.

Definition 1.4.
The division $(A_1, \dots, A_n; \pi)$ of the cake is called *fair* if $\mu_i(A_{\pi(i)}) \geq \frac{1}{n}$ for all i. It is called *envy-free* if $\mu_i(A_{\pi(i)}) \geq \mu_i(A_{\pi(j)})$ for all i, j.

The last condition says that each person receives a piece that is (according to its measure) at least as large as all the other pieces.

The permutation π might seem unnecessary at this point, but for the purpose of upcoming proofs we need to be able to assign the n pieces of a fixed division to the n people.

Practical Cake-Cutting

There are several interesting approaches to obtaining solutions to *fair cake-cutting*. The simplest is tl following. Let t be the smallest value such that there exists an i with $\mu_i([0,t]) = \frac{1}{n}$. This means that for all other j, the rest of the cake, i.e., $[t, 1]$, has size at least $\frac{n-1}{n}$. Therefore, person i is to receive the piece $[0, t]$ and the others proceed by induction on the rescaled piece. Note that this procedure produces $n - 1$ cuts of the cake, and hence n intervals, i.e., the fewest number possible.

This procedure is often referred to as the *moving-knife algorithm*, and it works as follows. Some person slowly moves a knife along the cake. If the portion of the cake that has been covered by the moving knife has reached size $\frac{1}{n}$ for some person, then this person yells "stop!" The cake is cut right there, and the person who yelled receives the piece. If more than one person yelled, the piece is giver to ~~one of them. From a practical viewpoint, this has the advantage that every person feels treated~~ fairly. But of course, in general, the divisions obtained in this way are not envy-free. There are severa algorithms one can use to obtain an envy-free division of the cake. But these often require many cuts of the cake; cf. [RW98].

The Simplex as Solution Space

Here we want to concentrate on the existence and approximation of an envy-free solution that can be obtained by $n-1$ cuts.

A division of the unit interval into n successive intervals is determined by the vector $(t_1, \dots, t_n)$ o their lengths, and all possible such division vectors constitute the standard $(n-1)$-simplex if we allov intervals of length zero.

The first proof of the existence of an envy-free fair-division solution with $n-1$ cuts by Woodall [Woo80] is a construction whose topological engine is Brouwer's fixed-point theorem. Ther is an easier construction that—not surprisingly—relies on Sperner's lemma. Moreover, this construction yields a method to find approximate solutions described by Su [Su99], which has a nice implementation called the "Fair division calculator" and is available on the Internet.

A Sperner Labeling Approach

In order to find an approximate solution, consider a barycentric subdivision $\mathrm{sd}^k \Delta^{n-1}$ for some k. Ou construction will consist of two consecutive labelings of the vertices, the second of which is going to be a Sperner labeling. The first labeling is rather simple. Label the vertices of $\mathrm{sd}^k \Delta^{n-1}$ with labels $p_1, \dots, p_n$ in such a manner that the vertices of each $(n-1)$-simplex obtain all labels $p_1, \dots, p_n$, as demonstrated in Fig. 1.8. Such a labeling is easy to derive and is the content of Exercise 4.

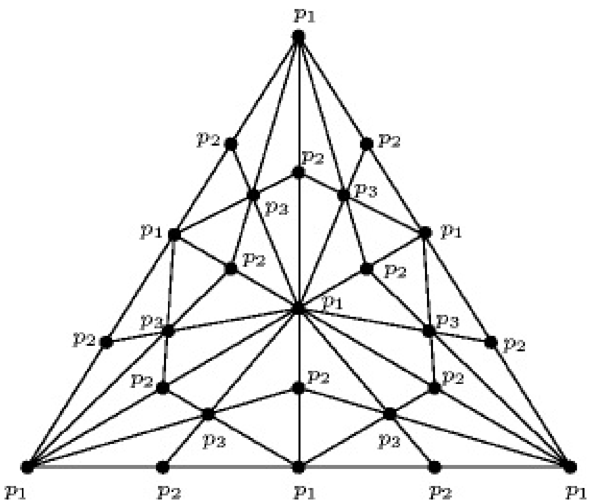


Fig. 1.8 First labeling of vert($\mathrm{sd}^k \Delta^{n-1}$)

To define the second labeling, consider a vertex $v = (t_1, \dots, t_n)$ of $\mathrm{sd}^k \Delta^{n-1}$ with, say, label p_{i_0}. I defines the division $I = [0, t_1] \cup [t_1, t_1 + t_2] \cup \cdots \cup [t_1 + \cdots + t_{n-1}, 1]$. Denote the kth interval by I_k and let $\mu(v) = \max\{\mu_{i_0}(I_1), \dots, \mu_{i_0}(I_n)\}$ be the size of the largest piece according to person i_0. Define the labeling by

$$\lambda(v) = \min\{j : \mu(I_j) = \mu(v)\}.$$

In other words, $\lambda(v)$ describes the number of a piece that is largest for person i_0. Certainly

$$\lambda(v) \in \{j : t_j \neq 0, j \in [n]\},$$

since the largest piece will not be an interval of length 0, and hence λ is a bona fide Sperner labeling. For an example see Fig. 1.9.

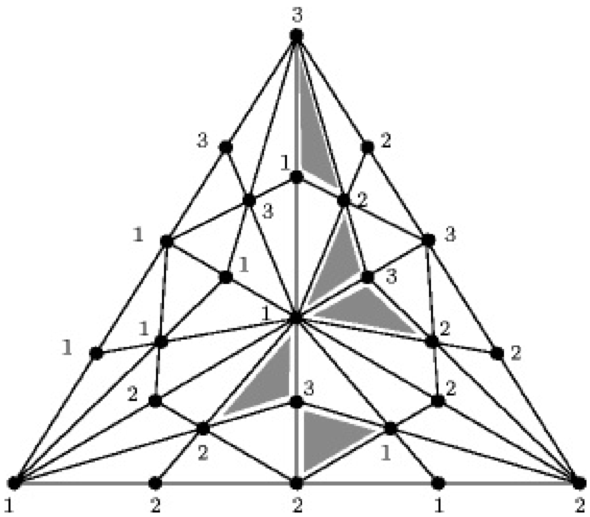


Fig. 1.9 An example of the labeling λ

By Sperner's lemma, we obtain a fully labeled $(n-1)$-simplex σ_k, which means that the n different people associated with the n vertices of σ_k all choose a different interval. More precisely, σ defines a permutation $\pi_k : [n] \to [n]$, where $\pi_k(i) = j$ if $\lambda(v) = j$ for the vertex v of σ_k labeled with p_i Now let x_k be the barycenter of the simplex σ_k and consider the sequence (x_k). By compactness, the exists a convergent subsequence (x_{i_k}). Since there is only a finite number of permutations of $[n]$, we may even choose this subsequence with the additional property that the sequence (π_{i_k}) of associated permutations is constant. Let $x = (t_1, \ldots, t_n)$ be the limit of this subsequence and π the constant permutation. The associated division $(A_1, \ldots, A_n; \pi)$ is the desired envy-free solution, as is easy to prove and is the content of an exercise. Thus we obtain the following result.

Theorem 1.5.
Let $\mu_1, \ldots, \mu_n$ be n continuous probability measures on the unit interval. Then there exists an envy-fre division $(A_1, \ldots, A_n; \pi)$ such that all of the A_i are intervals.

Note that, moreover, an approximate solution can be found in a finite number of steps: in fact, for an given $\varepsilon > 0$, there exist a $k \geq 0$, a simplex $\sigma_k \in \mathrm{sd}^k \Delta^{n-1}$, and a permutation π with the property that the division associated with the barycenter of σ_k together with π is envy-free up to an error of ε.

1.3 The Borsuk–Ulam Theorem and Tucker's Lemma

The Borsuk–Ulam theorem is a classical theorem in algebraic topology, and next to Brouwer's theorem, is one of the main results typically proven in an algebraic topology course to show the powe of homological methods. For some historical background on Stan Ulam and the history of the theorem, I recommend Gian-Carlo Rota's wonderful article [Rot87]. The Borsuk–Ulam theorem is often illustrated by the claim that at any moment in time, there is a pair of antipodal points on the surface of the earth with the same air pressure and temperature. We will present four versions of the theorem, most of which will play some role in the sequel. An illustration of the third version is given in Fig. 1.10.

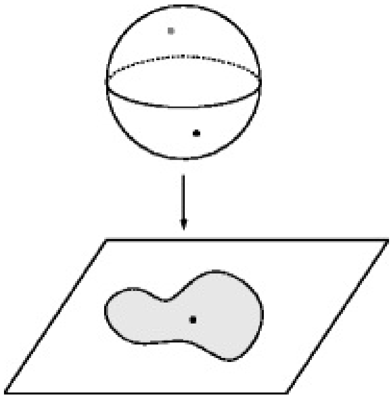


Fig. 1.10 A sphere made flat

Theorem 1.6 (Borsuk–Ulam).
The following statements hold.

1. *If $f:\mathbb{S}^n\to\mathbb{S}^m$ is a continuous antipodal map, i.e., f(−x) = −f(x) for all $x\in\mathbb{S}^n$, then n ≤ m.*
2. *If $f:\mathbb{S}^n\to\mathbb{R}^n$ is a continuous antipodal map, then there exists an $x\in\mathbb{S}^n$ such that f(x) = 0.*
3. *If $f:\mathbb{S}^n\to\mathbb{R}^n$ is a continuous map, then there exists $x\in\mathbb{S}^n$ such that f(x) = f(−x).*
4. *If $\mathbb{S}^n$ is covered by n + 1 subsets $S_1,\dots,S_{n+1}$ such that each of $S_1,\dots,S_n$ is open or closed, then one of the sets contains an antipodal pair of points, i.e., there exist an i ∈* [*n* + 1] and *$x\in\mathbb{S}^n$ such that $x,-x\in S_i$.*

Since Brouwer's fixed-point theorem is intimately related to Sperner's lemma,the same is true for the Borsuk–Ulam theorem and a lemma by Tucker. In thesequel, we will present a proof of the Borsuk–Ulam theorem by means of Tucker'slemma. For a proof using standard methods from algebraic topology, I recommendBredon [Bre93]. But in order to get used to the different ways the Borsuk–Ulamtheorem was stated, we will show that each of the four versions easily implies theothers.

Proof (of the equivalences).
(1 ⇒ 2) Assume $f:\mathbb{S}^n\to\mathbb{R}^n$ is an antipodal map without zero. Then the map

$$x\longmapsto\frac{f(x)}{||f(x)||}$$

is (by compactness of $\mathbb{S}^n$) a continuous antipodal map from $\mathbb{S}^n$ to $\mathbb{S}^{n-1}$, contradicting 1.

(2 ⇒ 3) Let $f:\mathbb{S}^n\to\mathbb{R}^n$ be a continuous map. Consider the continuous *and antipodal* map $g:\mathbb{S}^n\to\mathbb{R}^n$ defined by $g(x) = f(x) - f(-x)$. By statement 2., g has a zero, x, which yields the desired property for f.

(3 ⇒ 4) Let $\mathbb{S}^n$ be covered by $n + 1$ subsets $S_1, \dots, S_{n+1}$, such that each of $S_1, \dots, S_n$ is open or closed. Assume that none of $S_1, \dots, S_n$ contains an antipodal pair of points. Then $x \in S_i$ implies dist(

$-x, S_i) > 0$ for each $i \in [n]$ and $x \in \mathbb{S}^n$. We show this by considering separately the cases in which S_i is closed or open. If $A = S_i$ is closed and $x \in A$, then $-x \notin A$, and therefore $\mathrm{dist}(-x, A) > 0$. If $U = S_i$ is open and $x \in U$, then $\mathrm{dist}(x, \mathbb{S}^n \setminus U) > 0$, and since $-U \subseteq \mathbb{S}^n \setminus U$, we derive $\mathrm{dist}(-x, U) = \mathrm{dist}(x, -U) \geq \mathrm{dist}(x, \mathbb{S}^n \setminus U) > 0$.

We will now find an antipodal pair of points in S_{n+1} as follows. Consider the continuous map

$$\begin{aligned} f : \mathbb{S}^n &\to \mathbb{R}^n, \\ x &\longmapsto \begin{pmatrix} \mathrm{dist}(x, S_1) \\ \vdots \\ \mathrm{dist}(x, S_n) \end{pmatrix}. \end{aligned}$$

By assumption there exists an $x \in \mathbb{S}^n$ with $f(x) = f(-x)$. We are done if we can show that $x, -x \notin S_1 \cup \cdots \cup S_n$. We check this by showing $x, -x \notin S_i$ for each $i \in [n]$. If $\mathrm{dist}(x, S_i) = \mathrm{dist}(-x, S_i) >$, then clearly $x, -x \notin S_i$. If $\mathrm{dist}(x, S_i) = \mathrm{dist}(-x, S_i) = 0$, then $x, -x \notin S_i$ by the discussion above.

(4 ⇒ 1) Assume that there is an antipodal map $f : \mathbb{S}^n \to \mathbb{S}^{n-1}$. Now the important observation is that the $(n-1)$-dimensional sphere can be covered with $n+1$ closed sets, none of which contains an antipodal pair. To see this, consider an n-simplex in $\mathbb{R}^n$ with 0 in the interior. The radial projections $X_1, \ldots, X_{n+1}$ of the $n+1$ facets of dimension $n-1$ to the sphere yield the desired cover, as demonstrated in Fig. 1.11. Now let $S_i = f^{-1}(X_i)$. By continuity of f, the S_i are closed, and by the antipodality of f, they do not contain any antipodal pair of points.

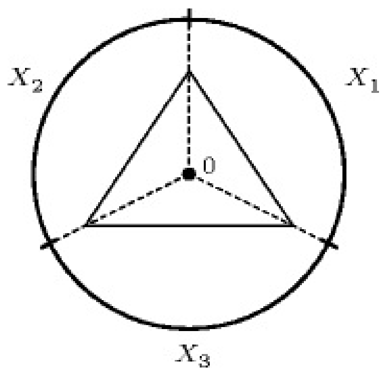


Fig. 1.11 A cover of the $(n-1)$-sphere with $n+1$ closed sets

For the last implication, note that the family, $X_1, \ldots, X_{n+1}$, of open sets $X_i = \{x \in \mathbb{S}^{n-1} \subseteq \mathbb{R}^n : x_i > 0\}$ for $i \in [n]$ and the closed set $X_{n+1} = \{x \in \mathbb{S}^{n-1} \subseteq \mathbb{R}^n : \forall i \in [n] : x_i \leq 0\}$ would have worked as well.

Tucker’s Lemma

To formulate Tucker’s lemma we will be interested in subdivisions of the n-dimensional sphere $\mathbb{S}^n$ that refine the “triangulation” of $\mathbb{S}^n$ by the coordinate hyperplanes. More precisely, we will consider subdivisions K of the boundary complex Γ^n of the cross polytope, which we will introduce now.

The $(n+1)$-dimensional *cross polytope* is defined to be the convex hull $Q^{n+1} = \mathrm{conv}\left(\{\pm e_1, \ldots, \pm e_{n+1}\}\right)$. Its boundary is the polyhedron of a geometric simplicial complex, whose simplices are given by the convex hulls of sets $\{\epsilon_{i_1} e_{i_1}, \ldots, \epsilon_{i_k} e_{i_k}\}$, not containing an antipodal pair $\pm e_j$. We denote the (geometric) simplicial complex given by this collection of geometric simplices by

$$\Gamma^n = \{\mathrm{conv}(\{\epsilon_{i_1}e_{i_1}, \dots, \epsilon_{i_k}e_{i_k}\}) : \quad 0 \le k \le n+1, \quad 1 \le i_1 < \cdots < i_k \le n+1, \epsilon_{i_j} \in \{\pm 1\}\}.$$

An alternative way to construct Γ^n is to take the $(n+1)$-fold join of two-point sets, i.e., 0-spheres

$$\Gamma^n = \{\pm e_1\} * \{\pm e_2\} * \cdots * \{\pm e_{n+1}\},$$

where $\{\pm e_i\}$ serves as an abbreviation of the geometric complex $\{\emptyset, +e_1, -e_1\}$. An illustration is given in Fig. 1.12.

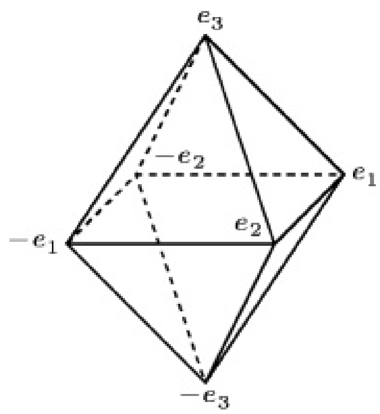


Fig. 1.12 The cross polytope Q^3

Tucker's lemma is concerned with *antipodally symmetric triangulations K*, i.e., triangulations wi the property that $\sigma \in K$ if and only if $-\sigma \in K$.

Lemma 1.7 (Tucker).
Let K be an antipodally symmetric subdivision of Γ^n, and $\lambda : \mathrm{vert}(K) \to \{\pm 1, \dots, \pm n\}$ *an antipodally symmetric labeling of K, i.e., $\lambda(-v) = -\lambda(v)$ for all v. Then there exists a complementary edge, i.e., an edge $uv \in K$ with $\lambda(u) + \lambda(v) = 0$.*

Since the triangulation K and the labeling are antipodally symmetric, we can sketch an example by just showing one side of the boundary of the cross polytope; cf. Fig. 1.13.

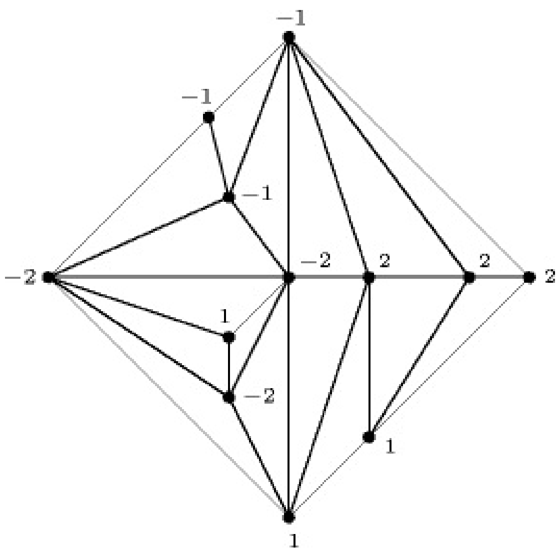


Fig. 1.13 A front side of a labeled subdivision of Γ^2

Tucker's lemma is an immediate corollary of the Borsuk–Ulam theorem. The proof of this is the content of Exercise 8. Conversely, the Borsuk–Ulam theorem may be derived from Tucker's lemma, as we show next. Afterwards we will be concerned with a combinatorial proof of Tucker's lemma tha also finds a complementary edge, in a manner analogous to the combinatorial proof of Sperner's lemma. The first such proof, given by Freund and Todd [FT81], even proves that there is always an odd number of complementary edges. We will present a more recent proof by Prescott and Su, and moreover give an elegant direct proof for the existence of a complementary edge.

Proof (of the Borsuk–Ulam theorem with the Tucker lemma).
Since $|\Gamma^n|$ and $\mathbb{S}^n$ are homeomorphic, we may assume for contradiction that there exists an antipoda map $f:|\Gamma^n| \to \mathbb{R}^n$ without a zero. Hence, there exists an $\varepsilon > 0$ such that $\|f\|_\infty \geq \epsilon$, i.e., for each $x \in \mathbb{S}^n$ there exists a coordinate i with $|f_i(x)| \geq \varepsilon$. By continuity of f, there exists a k such that for all edges u of $K = \mathrm{sd}^k \Gamma^n$, we have $\|f(u) - f(v)\|_\infty < \epsilon$. We construct an antipodally symmetric labeling $\lambda : \mathrm{vert}(K) \to \{\pm 1, \ldots, \pm n\}$ as follows. Let

$$i(v) = \min\{i : |f_i(v)| \geq \epsilon\},$$

and define

$$\lambda(v) = \begin{cases} +i(v) & \text{if } f_{i(v)}(v) \geq \epsilon, \\ -i(v) & \text{if } f_{i(v)}(v) \leq -\epsilon. \end{cases}$$

Now $|f_i(-v)| = |-f_i(v)| = |f_i(v)|$ implies $i(-v) = i(v)$, and hence $\lambda(-v) = -\lambda(v)$. By Tucker's lemma, there exists an edge uv in K such that for some $i \in [n]$, we have $\lambda(u) = +i$ and $\lambda(v) = -i$ (afte maybe switching u and v). Hence, by definition of the labeling, $f_i(u) \geq \varepsilon$ and $f_i(v) \leq -\varepsilon$, contradictin $\|f(u) - f(v)\|_\infty < \epsilon$.

1.4 A Generalization of Tucker's Lemma

As announced, we now turn our attention to combinatorics and consider a generalization of Tucker's lemma by Ky Fan [Fan52]. The idea is to decouple the size of the label set from the dimension n. We will be interested in certain *alternatingly* labeled simplices. Let K be an antipodally symmetric subdivision of Γ^n and $\lambda : \mathrm{vert}(K) \to \{\pm 1, \ldots, \pm m\}$ an antipodally symmetric labeling of K. A d-simple σ is called +-*alternating*, resp. −-*alternating*, if it has labels $\{+j_0, -j_1, +j_2, \ldots, (-1)^d j_d\}$, resp. $\{-j_0, +j_1, -j_2, \ldots, (-1)^{d+1} j_d\}$, where $1 \leq j_0 < j_1 < \cdots < j_d \leq m$. For an illustration we refer to Fig. 1.14.

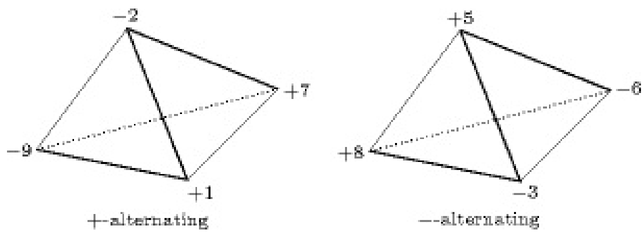


Fig. 1.14 A + - and a − -alternating 3-dimensional simplex

Theorem 1.8 (Ky Fan, weak version).

sample content of A Course in Topological Combinatorics

- read Prophecy: Child of Earth (Symphony of Ages, Book 2)
- **click Dangerous Women, Part II**
- click Pies and Tarts with Heart: Expert Pie-Building Techniques for 60+ Sweet and Savory Vegan Pies online
- *download online Big Data Analytics: Disruptive Technologies for Changing the Game*

- http://studystrategically.com/freebooks/Cooked--A-Natural-History-of-Transformation.pdf
- http://deltaphenomics.nl/?library/Command-and-Control--Nuclear-Weapons--the-Damascus-Accident--and-the-Illusion-of-Safety.pdf
- http://hasanetmekci.com/ebooks/Pies-and-Tarts-with-Heart--Expert-Pie-Building-Techniques-for-60--Sweet-and-Savory-Vegan-Pies.pdf
- http://www.mmastyles.com/books/Clairvoyance-and-Occult-Powers.pdf