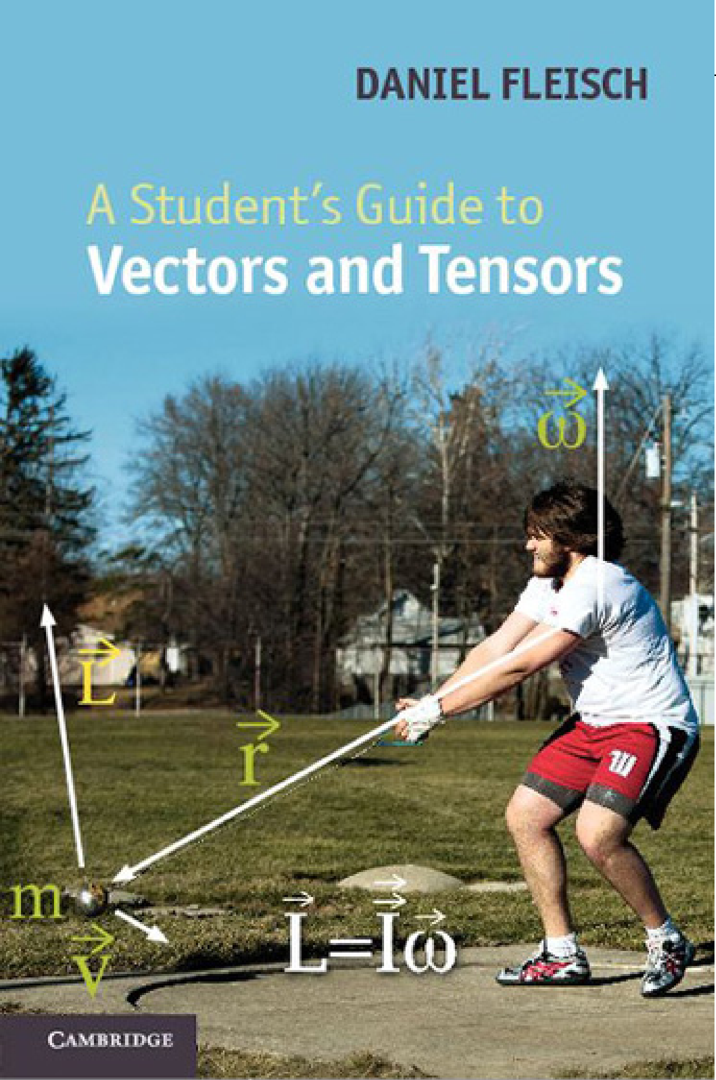
DANIEL FLEISCH
A Student's Guide to
Vectors and Tensors
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A Student's Guide to Vectors and Tensors

Vectors and tensors are among the most powerful problem-solving tools available, with applications ranging from mechanics and electromagnetics to general relativity. Understanding the nature and application of vectors and tensors is critically important to students of physics and engineering.

Adopting the same approach as in his highly popular *A Student's Guide to Maxwell's Equations*, Fleisch explains vectors and tensors in plain language. Written for undergraduate and beginning graduate students, the book provides a thorough grounding in vectors and vector calculus before transitioning through contra and covariant components to tensors and their applications. Matrices and their algebra are reviewed on the book's supporting website, which also features interactive solutions to every problem in the text, where students can work through a series of hints or choose to see the entire solution at once. Audio podcasts give students the opportunity to hear important concepts in the book explained by the author.

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A Student's Guide to Vectors and Tensors

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Preface

This book has one purpose: to help you understand vectors and tensors so that you can use them to solve problems. If you're like most students, you first encountered vectors when you took a course dealing with mechanics in high school or college. At that level, you almost certainly learned that vectors are mathematical representations of quantities that have both magnitude and direction, such as velocity and force. You may also have learned how to add vectors graphically and by using their components in the x-, y- and z-directions.

That's a fine place to start, but it turns out that such treatments only scratch the surface of the power of vectors. You can harness that power and make it work for you if you're willing to delve a bit deeper – to see vectors not just as objects with magnitude and direction, but rather as objects that behave in very predictable ways when viewed from different reference frames. That's because vectors are a subset of a larger class of objects called "tensors," which most students encounter much later in their academic careers, and which have been called "the facts of the Universe." It is no exaggeration to say that our understanding of the fundamental structure of the universe was changed forever when Albert Einstein succeeded in expressing his theory of gravity in terms of tensors.

I believe, and I hope you'll agree, that tensors are far easier to understand if you first establish a stronger foundation in vectors, one that can help you cross the bridge between the "magnitude and direction" level and the "facts of the Universe" level. That's why the first three chapters of this book deal with vectors, the fourth chapter discusses coordinate transformations, and the last two chapters discuss higher-order tensors and some of their applications.

One reason you may find this book helpful is that if you spend a few hours looking through the published literature and on-line resources for vectors and tensors in physics and engineering, you're likely to come across statements such as these:

"A vector is a mathematical representation of a physical entity characterized by magnitude and direction."

"A vector is an ordered sequence of values."

"A vector is a mathematical object that transforms between coordinate systems in certain ways."

"A vector is a tensor of rank one."

"A vector is an operator that turns a one-form into a scalar."

You should understand that every one of these definitions is correct, but whether it's useful to you depends on the problem you're trying to solve. And being able to see the relationship between statements like these should prove very helpful when you begin an in-depth study of subjects that use advanced vector and tensor concepts. Those subjects include Mechanics, Electromagnetism, General Relativity, and others.

As with most projects, a good first step is to make sure you understand the terminology that will be used to attack the problem. For that reason, Chapter 1 provides the basic definitions you'll need to begin understanding vectors and tensors. And if you're ready for more-advanced definitions, you can find those at the beginning of Chapter 5.

You may be wondering how this book differs from other texts that deal with vectors and/or tensors. Perhaps the most important difference is that approximately equal weight is given to vector and tensor concepts, with one entire chapter (Chapter 3) devoted to selected vector applications and another chapter (Chapter 6) dedicated to example tensor applications.

You'll also find the presentation to be very different from that of other books. The explanations in this book are written in an informal style in which mathematical rigor is maintained only insofar as it doesn't obscure the underlying physics. If you feel you already have a good understanding of vectors and may need only a quick review, you should be able to skim through Chapters 1 through 3 very quickly. But if you're a bit unclear on some aspects of vectors and how to apply them to problems, you may find these early chapters quite helpful. And if you've already seen tensors but are unsure of exactly what they are or how to apply them, then Chapters 4 through 6 may provide some insight.

As a student's guide, this book comes with two additional resources designed to help you understand and apply vectors and tensors: an interactive website and a series of audio podcasts. On the website, you'll find the complete solution to every problem presented in the text in interactive format – that means you'll be able to view the entire solution at once, or ask for a series of helpful hints that will guide you to the final answer. So when you see a statement in the text saying that you can learn more about something by looking at the end-of-chapter problems, remember that the full solution to every one of those problems is available to you. And if you're the kind of learner who benefits from hearing spoken words rather than just reading text, the audio podcasts are for you. These MP3 files walk you through each chapter of the book, pointing out important details and providing further explanations of key concepts.

Is this book right for you? It is if you're a science or engineering student and have encountered vectors or tensors in one of your classes, but you're not confident in your ability to apply them. In that case, you should read the book, listen to the accompanying podcasts, and work through the examples and problems before taking additional classes or a standardized exam in which vectors or tensors may appear. Or perhaps you're a graduate student struggling to make the transition from undergraduate courses and textbooks to the more-advanced material you're seeing in graduate school – this book may help you make that step.

And if you're neither an undergraduate nor a graduate student, but a curious young person or a lifelong learner who wants to know more about vectors, tensors, or their applications in Mechanics, Electromagnetics, and General Relativity, welcome aboard. I commend your initiative, and I hope this book helps you in your journey.

Acknowledgments

It was a suggestion by Dr. John Fowler of Cambridge University Press that got this book out of the starting gate, and it was his patient guidance and unflagging support that pushed it across the finish line. I feel very privileged to have worked with John on this project and on my *Student's Guide to Maxwell's Equations*, and I acknowledge his many contributions to these books. A project like this really does take a village, and many others should be recognized for their efforts. While pursuing her doctorate in Physics at Notre Dame University, Laura Kinnaman took time to carefully read the entire manuscript and made major contributions to the discussion of the Inertia tensor in Chapter 6. Wittenberg graduate Joe Fritchman also read the manuscript and made helpful suggestions, as did Carnegie-Mellon undergraduate Wyatt Bridgeman. Carrie Miller provided the perspective of a Chemistry student, and her husband Jordan Miller generously shared his LaTeX expertise. Professor Adam Parker of Wittenberg University and Daniel Ross of the University of Wisconsin did their best to steer me onto a mathematically solid foundation, and Professor Mark Semon of Bates College has gone far beyond the role of reviewer and deserves credit for rooting out numerous errors and for providing several of the better explanations in this work. I alone bear the responsibility for any remaining inconsistencies or errors.

I also wish to acknowledge all the students who have taken a class from me during the two years it took me to write this book. I very much appreciate their willingness to share their claim on my time with this project. The greatest sacrifice has been made by the unfathomably understanding Jill Gianola, who gracefully accommodated the expanding time and space requirements of my writing.

Contents

1

Vectors

1.1 Definitions (basic)

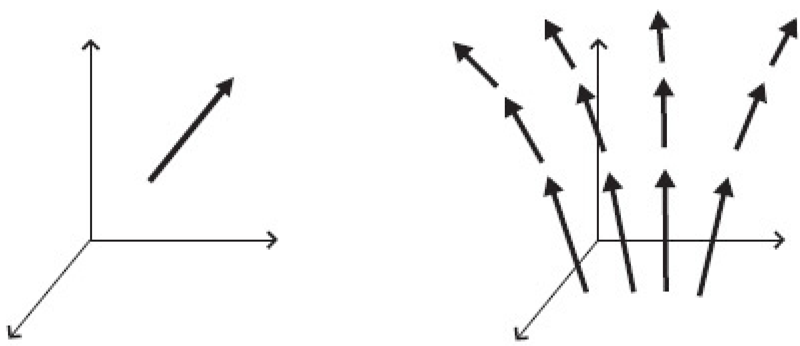
There are many ways to define a vector. For starters, here's the most basic:

> A vector is the mathematical representation of a physical entity that may be characterized by size (or "magnitude") and direction.

In keeping with this definition, speed (how fast an object is going) is not represented by a vector, but velocity (how fast and *in which direction* an object is going) does qualify as a vector quantity. Another example of a vector quantity is force, which describes how strongly and in what direction something is being pushed or pulled. But temperature, which has magnitude but no direction, is not a vector quantity.

The word "vector" comes from the Latin *vehere* meaning "to carry;" it was first used by eighteenth-century astronomers investigating the mechanism by which a planet is "carried" around the Sun.[1] In text, the vector nature of an object is often indicated by placing a small arrow over the variable representing the object (such as $\vec{F}$), or by using a bold font (such as $\boldsymbol{F}$), or by underlining (such as $\underline{F}$ or $\underset{\sim}{F}$). When you begin hand-writing equations involving vectors, it's very important that you get into the habit of denoting vectors using one of these techniques (or another one of your choosing). The important thing is not *how* you denote vectors, it's that you don't simply write them the same way you write non-vector quantities.

A vector is most commonly depicted graphically as a directed line segment or an arrow, as shown in Figure 1.1(a). And as you'll see later in this section, a vector may also be represented by an ordered set of N numbers, where N is the number of dimensions in the space in which the vector resides.



(a) (b)

Figure 1.1 Graphical depiction of a vector (a) and a vector field (b).

Of course, the true value of a vector comes from knowing what it represents. The vector in Figure 1.1(a), for example, may represent the velocity of the wind at some location, the acceleration of a rocket, the force on a football, or any of the thousands of vector quantities that you encounter in the world every day. Whatever else you may learn about vectors, you can be sure that every one of them has two things: size and direction. The magnitude of a vector is usually indicated by the length of the arrow, and it tells you the amount of the quantity represented by the vector. The scale is up to you (or whoever's drawing the vector), but once the scale has been established, all other vectors should be drawn to the same scale. Once you know that scale, you can determine the magnitude of any vector just by finding its length. The direction of the vector is usually given by indicating the angle between the arrow and one or more specified directions (usually the "coordinate axes"), and it tells you which way the vector is pointing.

So if vectors are characterized by their magnitude and direction, does that mean that two equally long vectors pointing in the same direction could in fact be considered to be the same vector? In other words, if you were to move the vector shown in Figure 1.1(a) to a different location without varying its length or its pointing direction, would it still be the same vector? In some applications, the answer is "yes," and those vectors are called free vectors. You can move a free vector anywhere you'd like as long as you don't change its length or direction, and it remains the same vector. But in many physics and engineering problems, you'll be dealing with vectors that apply *at a given location*; such vectors are called "bound" or "anchored" vectors, and you're not allowed to relocate bound vectors as you can free vectors.[2] You may see the term "sliding" vectors used for vectors that are free to move along their length but are not free to change length or direction; such vectors are useful for problems involving torque and angular motion.

You can understand the usefulness of bound vectors if you think about an application such as representing the velocity of the wind at various points in the atmosphere. To do that, you could choose to draw a bound vector at each point of interest, and each of those vectors would show the speed and direction of the wind at that location (most people draw the vector with its tail – the end without the arrow – at the point to which the vector is bound). A collection of such vectors is called a vector field; an example is shown in Figure 1.1(b).

If you think about the ways in which you might represent a bound vector, you may realize that the vector can be defined simply by specifying the start and end points of the arrow. So in a three-dimensional Cartesian coordinate system, you only need to know the values of x, y, and z for each end of the vector, as shown in Figure 1.2(a) (you can read about vector representation in non-Cartesian coordinate systems later in this chapter).

Now consider the special case in which the vector is anchored to the origin of the coordinate system (that is, the end without the arrowhead is at the point of intersection of the coordinate axes, as shown in Figure 1.2(b).[3] Such vectors may be completely specified simply by listing the three numbers that represent the x-, y-, and z-coordinates of the vector's end point. Hence a vector anchored to the origin and stretching five units along the x-axis may be represented as (5,0,0). In this representation, the values that represent the vector are called the "components" of the vector, and the number of components it takes to define a vector is equal to the number of dimensions in the space in which the vector exists. So in a two-dimensional space a vector may be represented by a pair of numbers, and in four-dimensional spacetime vectors may appear as lists of four numbers. This explains why a horizontal list of numbers is called a "row vector" and a vertical list of numbers is called a "column vector" in computer science. The number of values in such vectors tells you how many dimensions there are in the space in which the vector resides.

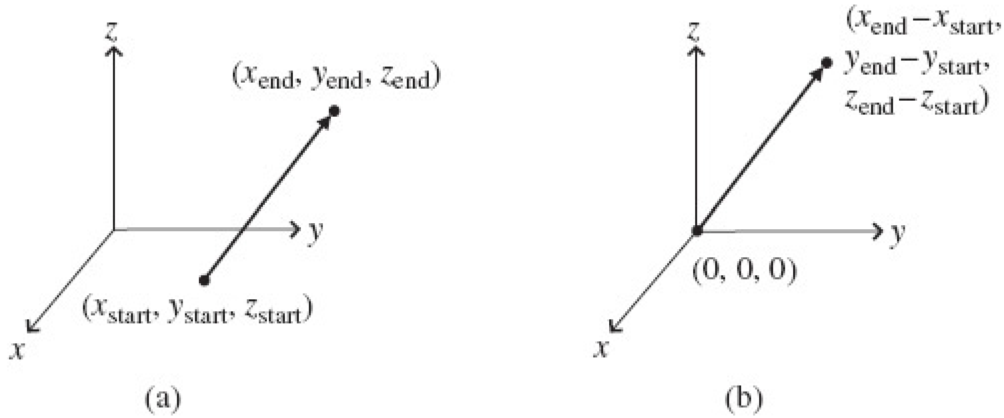


Figure 1.2 A vector in 3-D Cartesian coordinates.

To understand how vectors are different from other entities, it may help to consider the nature of some things that are clearly *not* vectors. Think about the temperature in the room in which you're sitting – at each point in the room, the temperature has a value, which you can represent by a single number. That value may well be different from the value at other locations, but at any given point the temperature can be represented by a single number, the magnitude. Such magnitude-only quantities have been called "scalars" ever since W.R. Hamilton referred to them as "all values contained on the one scale of progression of numbers from negative to positive infinity."[4] Thus

> A scalar is the mathematical representation of a physical entity that may be characterized by magnitude only.

Other examples of scalar quantities include mass, charge, energy, and speed (defined as the magnitude of the velocity vector). It is worth noting that the *change* in temperature over a region of space does have both magnitude and direction and may therefore be represented by a vector, so it's possible to produce vectors from groups of scalars. You can read about just such a vector (called the "gradient" of a scalar field) in Chapter 2.

Since scalars can be represented by magnitude only (single numbers) and vectors by magnitude and direction (three numbers in three-dimensional space), you might suspect that there are other entities involving magnitude and directions that are more complex than vectors (that is, requiring more numbers than the number of spatial dimensions). Indeed there are, and such entities are called "tensors."[5] You can read about tensors in the last three chapters of this book, but for now this simple definition will suffice:

> A tensor is the mathematical representation of a physical entity that may be characterized by magnitude and multiple directions.

An example of a tensor is the inertia that relates the angular velocity of a rotating object to its angular momentum. Since the angular velocity vector has a direction and the angular momentum vector has a (potentially different) direction, the inertia tensor involves multiple directions.

And just as a scalar may be represented by a single number and a vector may be represented by a sequence of three numbers in 3-dimensional space, a tensor may be represented by an array of 3^R numbers in 3-dimensional space. In this expression, "R" represents the rank of the tensor. So in 3-dimensional space, a second-rank tensor is represented by $3^2 = 9$ numbers. In N-dimensional space, scalars still require only one number, vectors require N numbers, and tensors require N^R numbers.

Recognizing scalars, vectors, and tensors is easy once you realize that a scalar can be represented by a single number, a vector by an ordered set of numbers, and a tensor by an array of numbers. So in three-dimensional space, they look like this:

Scalar	Vector	Tensor (Rank 2)
(x)	(x_1, x_2, x_3) or $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$	$\begin{pmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{pmatrix}$

Note that scalars require no subscripts, vectors require a single subscript, and tensors require two or more subscripts – the tensor shown here is a tensor of rank 2, but you may also encounter higher-rank tensors, as discussed in Chapter 5. A tensor of rank 3 may be represented by a three-dimensional array of values.

With these basic definitions in hand, you're ready to begin considering the ways in which vectors can be put to use. Among the most useful of all vectors are the Cartesian unit vectors, which you can read about in the next section.

1.2 Cartesian unit vectors

If you hope to use vectors to solve problems, it's essential that you learn how to handle situations involving more than one vector. The first step in that process is to understand the meaning of special vectors called "unit vectors" that often serve as markers for various directions of interest (unit vectors may also be called "versors").

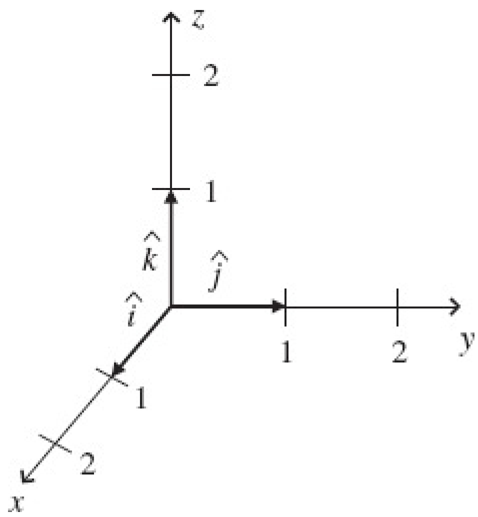


Figure 1.3 Unit vectors in 3-D Cartesian coordinates.

The first unit vectors you're likely to encounter are the unit vectors $\hat{x}$, $\hat{y}$, $\hat{z}$ (also called $\hat{\imath}$, $\hat{\jmath}$, $\hat{k}$) that point in the direction of the *x*-, *y*-, and *z*-axes of the three-dimensional Cartesian coordinate system, as shown in Figure 1.3. These vectors are called unit vectors because their length (or magnitude) is always exactly equal to unity, which is another name for "one." One what? One of whatever units you're using for that axis.

You should note that the Cartesian unit vectors $\hat{\imath}, \hat{\jmath}, \hat{k}$ can be drawn at any location, not just at the origin of the coordinate system. This is illustrated in Figure 1.4. As long as you draw a vector of unit length pointing in the same direction as the direction of the (increasing) *x*-axis, you've drawn the $\hat{\imath}$ unit vector. So the Cartesian unit vectors show you the directions of the *x*, *y*, and *z* axes, *not* the location of the origin.

As you'll see in Chapter 2, unit vectors can be extremely helpful when doing certain operations such as specifying the portion of a given vector pointing in a certain direction. That's because unit vectors don't have their own magnitude to throw into the mix (actually, they do have their own magnitude, but it is always one).

So when you see an expression such as "$5\hat{\imath}$," you should think "5 units along the positive *x*-direction." Likewise, $-3\hat{\jmath}$ refers to 3 units along the negative *y*-direction, and $\hat{k}$ indicates one unit along the positive *z*-direction.

Of course, there are other coordinate systems in addition to the three perpendicular axes of the Cartesian system, and unit vectors exist in those coordinate systems as well; you can see some examples in Section 1.5. One advantage of the Cartesian unit vectors is that they point in the same direction no matter where you go; the *x*-, *y*-, and *z*-axes run in straight lines all the way out to infinity, and the Cartesian unit vectors are parallel to the directions of those lines everywhere.

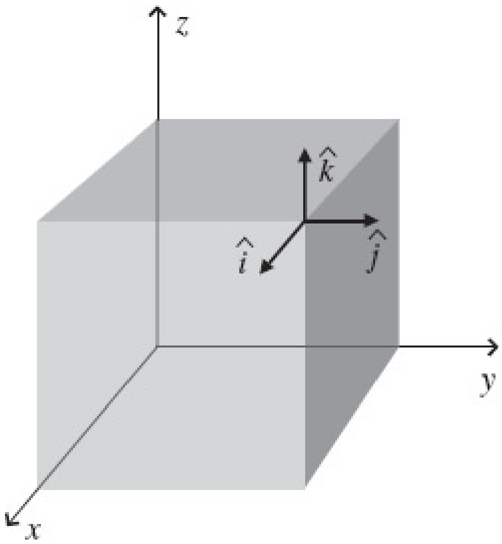


Figure 1.4 Cartesian unit vectors at an arbitrary point.

To put unit vectors such as $\hat{\imath}, \hat{\jmath}, \hat{k}$ to work, you need to understand the concept of vector components. The next section shows you how to represent vectors using unit vectors and vector components.

1.3 Vector components

The unit vectors described in the previous section are especially useful when they become part of the "components" of a vector. And what are the components of a vector? Simply stated, they are the pieces that can be used to make up the vector.

To understand vector components, think about the vector $\vec{A}$ shown in Figure 1.5. This is a bound vector, anchored at the origin and extending to the point (x = 0, y = 3, z = 3) in a three-dimensional

Cartesian coordinate system. So if you consider the coordinate axes as representing the corner of a room, this vector is embedded in the back wall (the *yz* plane).

Imagine you're trying to get from the beginning of vector $\vec{A}$ to the end – the direct route would be simply to move in the direction of the vector. But if you were constrained to move only in the directions of the axes, you could get from the origin to your destination by taking three (unit) steps along the *y*-axis, then turning 90° to your left, and then taking three more (unit) steps in the direction of the *z*-axis.

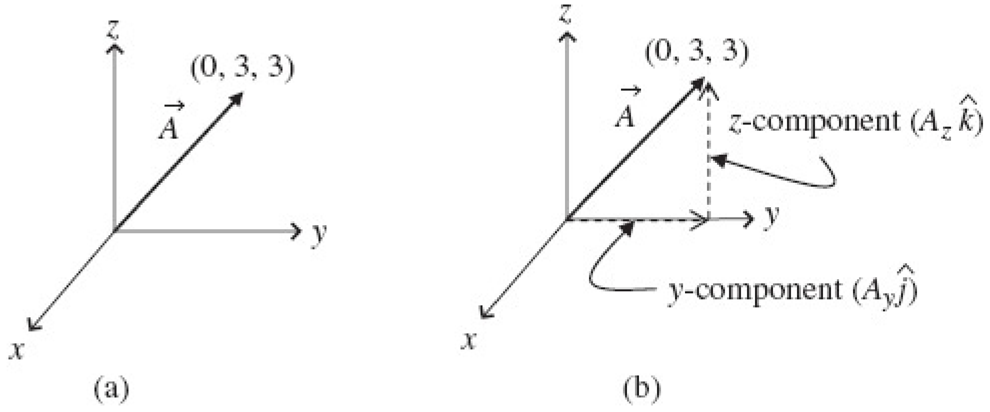


Figure 1.5 vector $\vec{A}$ and its components.

What does this little journey have to do with the components of a vector? Simply this: the lengths of the components of vector $\vec{A}$ are the distances you traveled in the directions of the axes. Specifically, in this case the magnitude of the *y*-component of vector $\vec{A}$ (written as A_y) is just the distance you traveled in the direction of the *y*-axis (3 units), and the magnitude of the *z*-component of vector $\vec{A}$ (written as A_z) is the distance you traveled in the direction of the *z*-axis (also 3 units). Since you didn't move at all in the direction of the *x*-axis, the magnitude of the *x*-component of vector $\vec{A}$ (written as A_x) is zero.

A very handy and compact way of writing a vector as a combination of vector components is this:

$$\vec{A} = A_x\hat{\imath} + A_y\hat{\jmath} + A_z\hat{k}, \tag{1.1}$$

where the magnitudes of the vector components (A_x, A_y, and A_z) tell you how many unit steps to take in each direction ($\hat{\imath}$, $\hat{\jmath}$, and $\hat{k}$) to get from the beginning to the end of vector $\vec{A}$.[6]

When you read about vectors and vector components, you're likely to run across statements such as "The components of a vector are the projections of the vector onto the coordinate axes." As you can see in Chapter 4, exactly how those projections are made can have a significant influence on the nature of the components you get. But in Cartesian coordinate systems (and other "orthogonal" systems in which the axes are perpendicular to one another), the concept of projection onto the coordinate axes is unambiguous and may be very helpful in picturing the components of a vector.

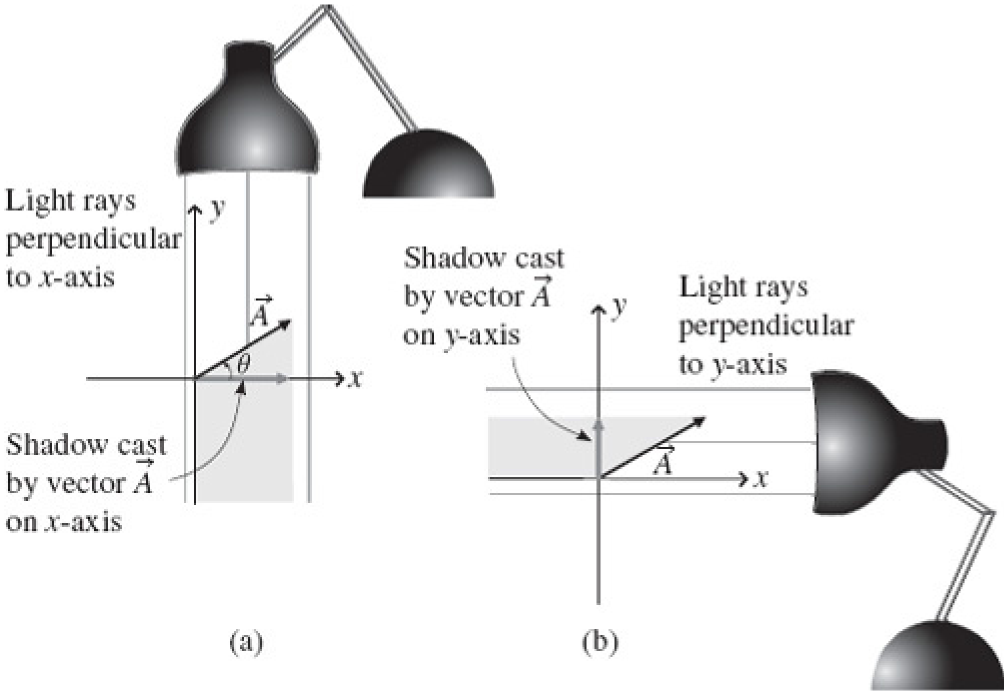


Figure 1.6 Vector components as projections onto *x*-and *y*-axes.

To understand how this works, take a look at vector $\vec{A}$ and the light sources and shadows in Figure 1.6. As you can see in Figure 1.6(a), the direction of the light that produces the shadow on the *x*-axis is parallel to the *y*-axis (actually antiparallel since it's moving in the negative *y*-direction), which in this case is the same as saying that the direction of the light is perpendicular to the *x*-axis.

Likewise, in Figure 1.6(b), the direction of the light that produces the shadow on the *y*-axis is antiparallel to the *x*-axis, which is of course perpendicular to the *y*-axis. This may seem like a trivial point, but when you encounter non-orthogonal coordinate systems, you'll find that the direction parallel to one axis is not necessarily perpendicular to another axis, which gives rise to an entirely different type of vector component. This simple fact has profound implications for the behavior of vectors and tensors for observers in different reference frames, as you'll see in Chapters 4, 5, and 6.

No such issues arise in the two-dimensional Cartesian coordinate system shown in Figure 1.6, and in this case the magnitudes of the components of vector $\vec{A}$ are easy to determine. If the angle between vector $\vec{A}$ and the positive *x*-axis is θ, as shown in Figure 1.6a, it's clear that the length of $\vec{A}$ can be seen as the hypotenuse of a right triangle. The sides of that triangle along the *x*- and *y*-axes are the components A_x and A_y. Hence by simple trigonometry you can write:

$$A_x = |\vec{A}| \cos(\theta),$$
$$A_y = |\vec{A}| \sin(\theta), \tag{1.2}$$

where the vertical bars on each side of $\vec{A}$ signify the magnitude (length) of vector $\vec{A}$. Notice that so long as you measure the angle θ *from the positive x-axis* in the direction toward the positive *y*-axis (that is, counterclockwise in this case), these equations will give the correct sign for the *x*- and *y*-components no matter which quadrant the vector occupies.

For example, if vector $\vec{A}$ is a vector with a length of 7 meters pointing in a direction 210° counter-clockwise from the +*x*-axis, the *x*- and *y*-components are given by Eq. 1.2 as

$$A_x = |\vec{A}|\cos(\theta) = 7\text{m}\cos 210° = –6.1\text{ m},$$
$$A_y = |\vec{A}|\sin(\theta) = 7\text{m}\sin 210° = –3.5\text{m}. \tag{1.3}$$

As expected for a vector pointing down and to the left from the origin, both components are negative.

It's equally straightforward to find the length and direction of a vector if you're given the vector's Cartesian components. Since the vector forms the hypotenuse of a right triangle with sides A_x and A_y, the Pythagorean theorem tells you that the length of $\vec{A}$ must be

$$|\vec{A}| = \sqrt{A_x^2 + A_y^2}, \tag{1.4}$$

and from trigonometry

$$\theta = \arctan\left(\frac{A_y}{A_x}\right), \tag{1.5}$$

where θ is measured counter-clockwise from the positive x-axis in a right-handed coordinate system. If you try this with the components of vector $\vec{A}$ from Eq. 1.3 and end up with a direction of 30° rather than 210°, remember that unless you have a four-quadrant arctan function on your calculator, you must add 180° to the angle whenever the denominator of the expression (A_x in this case) is negative.

Once you have a working understanding of unit vectors and vector components, you're ready to do basic vector operations. The entirety of Chapter 2 is devoted to such operations, but two of them are needed for the remainder of this chapter. For that reason, you can read about vector addition and multiplication by a scalar in the next section.

1.4 Vector addition and multiplication by a scalar

If you've read the previous section on vector components, you've already seen two vector operations in action. Those two operations are the addition of vectors and multiplication of a vector by a scalar. Both of these operations are used in the expansion of a vector in terms of vector components as in Eq. 1.1 from Section 1.3:

$$\vec{A} = A_x\hat{\imath} + A_y\hat{\jmath} + A_z\hat{k}.$$

In each of these terms, the unit vector ($\hat{\imath}$, $\hat{\jmath}$, or $\hat{k}$) is being multiplied by a scalar (A_x, A_y, or A_z), and you already know the effect of that: it produces a new vector, in the same direction as the unit vector, but longer than unity by the value of the component (or shorter if the magnitude of the component is between zero and one). So multiplying a vector by any positive scalar does not change the direction of the vector, but only scales the length of the vector. Hence, $5\vec{A}$ is a vector in exactly the same direction as $\vec{A}$, but with length five $\vec{A}$, as shown in Figure 1.7(a). Likewise, multiplying $\vec{A}$ by (1/2) produces a vector that points in the same direction as $\vec{A}$ but is only half as long. So the vector component $A_x\hat{\imath}$ is a vector in the $\hat{\imath}$ direction, but with length A_x units (since $\hat{\imath}$ has a length of one unit).

There is a caveat that goes with the "changes length, not direction" rule when multiplying a vector by a scalar: if the scalar is *negative*, then the vector is reversed in direction in addition to being scaled in length. Thus multiplying vector $\vec{B}$ by –2 produces the new vector $-2\vec{B}$, and that vector is twice as long as $\vec{B}$, as shown in Figure 1.7(b).

The other operation going on in Eq. 1.1 is vector addition, and you already have an idea of what that means if you recall Figure 1.5 and the process of getting from the beginning of vector $\vec{A}$ to the end. In that process, the quantity $A_y\hat{j}$ represented not only the number of steps you took, but also the direction in which you took them. Likewise, the quantity $A_z\hat{k}$ represented the number of steps you took *in a different direction*. The fact that these two quantities include directional information means that you cannot simply add them together algebraically; you must add them "as vectors."

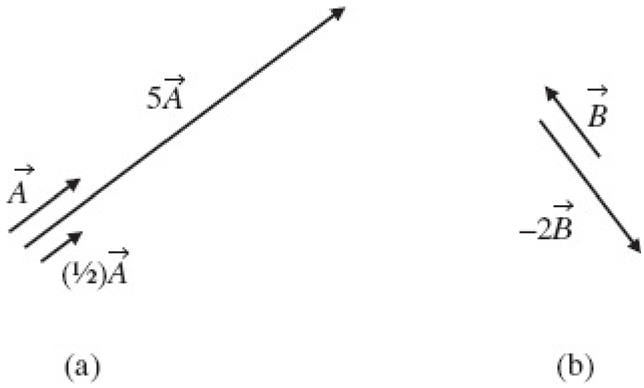


Figure 1.7 Multiplication of a vector by a scalar.

To accomplish vector addition graphically, you simply imagine moving one vector (without changing its length or direction) so that its tail is at the head of the other vector. The sum is then determined by making a new vector that begins at the start of the first vector and terminates at the end of the second vector. You can do this graphically, as in Figure 1.5(b), where the tail of vector $A_z\hat{k}$ is placed at the head of vector $A_y\hat{j}$, and the sum is the vector from the beginning of $A_y\hat{j}$ to the end of $A_z\hat{k}$.

This graphical "head-to-tail" approach to vector addition works for any vectors (and any number of vectors), not just two vectors that are perpendicular to one another (as $A_y\hat{j}$ and $A_z\hat{k}$ were). An example of this is shown in Figure 1.8. To graphically add the two vectors $\vec{A}$ and $\vec{B}$ in Figure 1.8(a), you simply imagine moving one of the two vectors so that its tail is at the position of the other vector's head (it doesn't matter which vector you choose to move; the result will be the same). This is illustrated in Figure 1.8(b), in which vector $\vec{B}$ has been displaced so that its tail is at the head of vector $\vec{A}$. The sum of these two vectors (called the "resultant" vector $\vec{C} = \vec{A} + \vec{B}$) is the vector that extends from the beginning of $\vec{A}$ to the end of $\vec{B}$.

Knowing how to add vectors graphically means you can always determine the sum of two or more vectors simply using a ruler and a protractor; just draw the vectors head-to-tail (being careful to maintain each vector's length and angle), sketch the resultant from the beginning of the first to the end of the last, and then measure the length (using the ruler) and angle (using the protractor) of the resultant. This approach can be both tedious and inaccurate, so here's an alternative approach that uses the components of each vector: if vector $\vec{C}$ is the sum of two vectors $\vec{A}$ and $\vec{B}$, then the magnitude of the *x*-component of vector $\vec{C}$ (which is just C_x) is the sum of the magnitudes of the *x*-components of vectors $\vec{A}$ and $\vec{B}$ (that is, $A_x + B_x$), and the magnitude of the *y*-component of vector

$\vec{C}$ (called C_y) is the sum of the magnitudes of the y-components of vectors $\vec{A}$ and $\vec{B}$ (that is, $A_y + B_y$). Thus

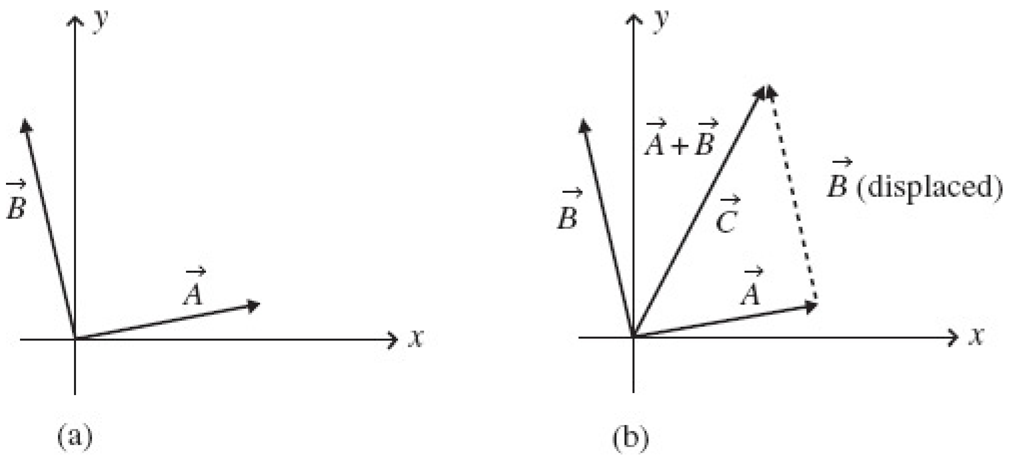


Figure 1.8 Graphical addition of vectors.

$$C_x = A_x + B_x\,,$$

$$C_y = A_y + B_y. \tag{1.6}$$

The rationale for this is shown in Figure 1.9.

Once you have the components C_x and C_y of the resultant vector $\vec{C}$, you can find the magnitude and direction of $\vec{C}$ using

$$|\vec{C}| = \sqrt{C_x^2 + C_y^2} \tag{1.7}$$

and

$$\theta = \arctan\left(\frac{C_y}{C_x}\right) \tag{1.8}$$

To see how this works in practice, imagine that vector $\vec{A}$ in Figure 1.9 is given by $\vec{A} = 6\hat{\imath} + \hat{\jmath}$ and vector $\vec{B}$ is given by $\vec{B} = -2\hat{\imath} + 8\,\hat{\jmath}$. To add these two vectors algebraically, you simply use Eqs. 1.6:

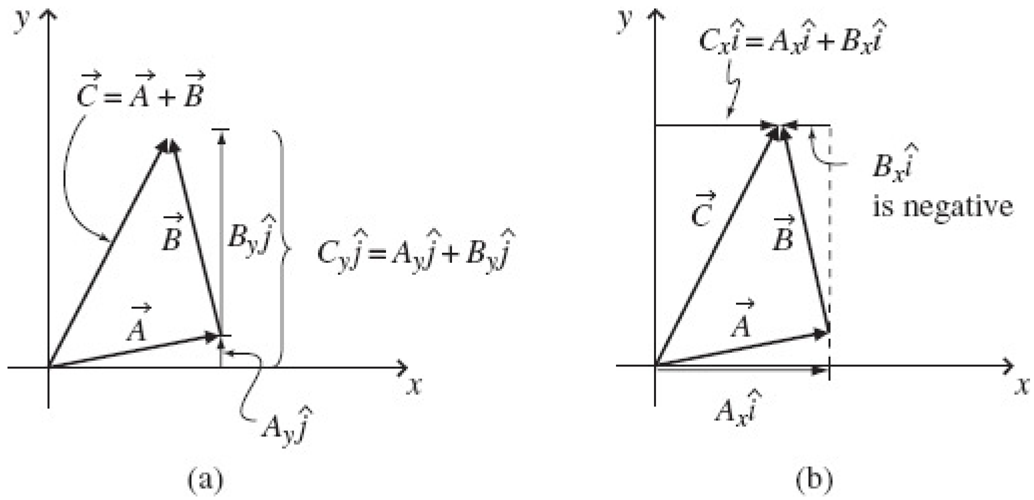


Figure 1.9 Component addition of vectors.

$$C_x = A_x + B_x = 6 + (-2) = 4,$$

$$C_y = A_y + B_y = 1 + 8 = 9,$$

so $\vec{C} = 4\hat{\imath} + 9\hat{\jmath}$. If you wish to know the magnitude of $\vec{C}$, you can just plug the components into Eq. 1.7 to get

$$\begin{aligned} |\vec{C}| &= \sqrt{C_x^2 + C_y^2} = \sqrt{4^2 + 9^2} \\ &= \sqrt{16 + 81} = 9.85. \end{aligned}$$

And the angle that $\vec{C}$ makes with the positive x-axis is given by Eq. 1.8:

$$\begin{aligned} \theta &= \arctan\left(\frac{C_y}{C_x}\right) \\ &= \arctan\left(\frac{9}{4}\right) = 66.0°. \end{aligned}$$

With the basic operations of vector addition and multiplication of a vector by a scalar in hand, you're ready to begin thinking about the more advanced uses of vectors. But you're also ready to attack a variety of problems involving vectors, and you can find a set of such problems at the end of this chapter.[7]

1.5 Non-Cartesian unit vectors

The three straight, mutually perpendicular axes of the Cartesian coordinate system are immensely useful for a variety of problems in physics and engineering. Some problems, however, are much easier to solve in other coordinate systems, often because the axes of those systems more closely align with the directions over which one or more of the parameters relevant to the problem remain constant or vary in a predictable manner. The unit vectors of such non-Cartesian coordinate systems are the subject of this section, and transformations between coordinate systems are discussed in Chapter 4.

As described earlier, it takes exactly N numbers to unambiguously represent any location in a space of N dimensions, which means you have to specify three numbers (such as *x*, *y*, and *z*) to designate a location in our Universe of three spatial dimensions. However, on the two-dimensional surface of the Earth (ignoring height variation for the moment) it takes only two numbers (latitude and longitude, for example) to designate a specific point. And one of the few benefits to living on a long, infinitely thin island is that you can set up a rendezvous using only a single number to describe the location ("I'll be waiting for you at 3.75 kilometers").

Of course, numbers define locations only after you've defined the *coordinate system* that you're using. For example, do you mean 3.75 kilometers from the east end of the island or from the west end? In every space of 1, 2, 3, or more dimensions, you can devise an infinite number of coordinate systems to specify locations in that space. In each of those coordinate systems, at each location there's one direction in which one of the coordinates is increasing the fastest, and if you lay a vector with length of one unit in that direction, you've defined a coordinate unit vector for that system. So in the Cartesian coordinate system, the $\hat{\imath}$ unit vector shows you the direction in which the *x*-coordinate increases, the $\hat{\jmath}$ unit vector shows you the direction in which the *y*-coordinate increases, and the $\hat{k}$ unit vector shows you the direction in which the *z*-coordinate increases. Other coordinate systems have their own coordinate unit vectors, as well.

Consider the two-dimensional coordinate systems shown in Figure 1.10. In a two-dimensional space, you know that it takes two numbers to specify any location, and those numbers could be *x* and *y*, defined along two straight axes that intersect at a right angle. The *x* value tells you how far you are to the right of the *y*-axis (or to the left if the *x* value is negative), and the *y* value tells you how far you are above the *x*-axis (or below if the *y* value is negative). But you could equally well specify any location in this two-dimensional space by noting how far and in what direction you've moved from the origin. In the standard version of these "polar" coordinates, the distance from the origin is called *r* and the direction is specified by giving the angle θ measured counterclockwise from the positive *x*-axis.

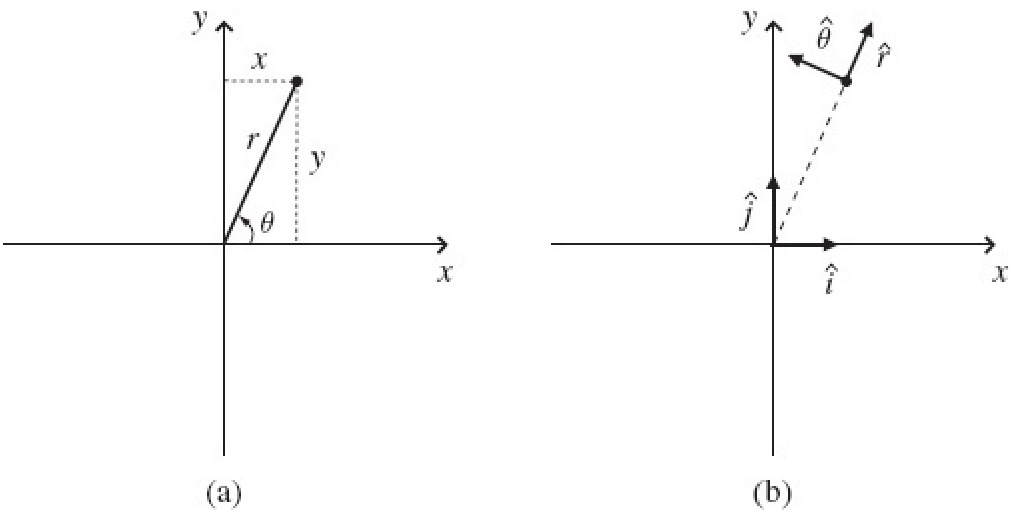


Figure 1.10 2-D rectangular (a) and polar (b) coordinates.

It's easy enough to figure out one set of coordinates if you know the others; for example, if you know the values of *x* and *y*, you can find *r* and θ using

$$
\begin{aligned}
r &= \sqrt{x^2 + y^2} \\
\theta &= \arctan\left(\frac{y}{x}\right).
\end{aligned} \tag{1.9}
$$

Likewise, if you have the values of r and θ, you can find x and y using

$$
\begin{aligned}
x &= r\cos(\theta) \\
y &= r\sin(\theta).
\end{aligned} \tag{1.10}
$$

For the point shown in Figure 1.10, if the values of x and y are 4 cm and 9 cm, then r has a value of approximately 9.85 cm and θ has a value of 66.0°. Clearly, whether you write (x, y) = (4cm, 9cm) or (r, θ) = (9.85 cm, 66.0°), you're referring to the same location; it's not the point that's changed, it's only the point's coordinates that are different.

And if you choose to use the polar coordinate system to represent the point, do unit vectors exist that serve the same function as $\hat{\imath}$ and $\hat{\jmath}$ in Cartesian coordinates? They certainly do, and with a little logic you can figure out which direction they must point. After all, you know that the unit vector $\hat{\imath}$ shows you the direction of increasing x and the unit vector $\hat{\jmath}$ shows you the direction of increasing y, but now you're using r and θ instead of x and y. So it seems reasonable that the unit vector $\hat{r}$ at any location should point in the direction of increasing r, and the unit vector $\hat{\theta}$ should point in the direction of increasing θ. For the point shown in Figure 1.10, that means that $\hat{r}$ should point up and to the right, in the direction of increasing r if θ is held constant. At that same point, $\hat{\theta}$ should point up and to the left, in the direction of increasing θ if r is held constant. These polar unit vectors are shown for one point in Figure 1.10(b).

An important consequence of this definition is that the directions of $\hat{r}$ and $\hat{\theta}$ will be different at different locations. They'll always be perpendicular to one another, but they will not point in the same directions as they do for the point in Figure 1.10. The dependence of the polar unit vectors on position can be seen in the following relations:

$$
\begin{aligned}
\hat{r} &= \cos(\theta)\hat{\imath} + \sin(\theta)\hat{\jmath} \\
\hat{\theta} &= -\sin(\theta)\hat{\imath} + \cos(\theta)\hat{\jmath}.
\end{aligned} \tag{1.11}
$$

So if $\theta = 0$ (which means your location is on the $+x$-axis), then $\hat{r} = \hat{\imath}$ and $\hat{\theta} = \hat{\jmath}$. But if $\theta = 90°$ (so your location is on the $+y$-axis), then $\hat{r} = \hat{\jmath}$ and $\hat{\theta} = -\hat{\imath}$.

Does this dependence on position mean that these unit vectors are not "real" vectors? That depends on your definition of a real vector. If you define a vector as a quantity with magnitude and direction, the polar unit vectors do meet your definition. But they do not meet the definition of free vectors described in Section 1.1, since they may not be moved without changing their direction.

This means that if you express a vector in polar coordinates and then take the derivative of that vector, you'll have to account for the change in the unit vectors, as well. That's one of the advantages offered by Cartesian coordinates – the unit vectors do not change no matter where you go in the space.

As you might expect, the situation is slightly more complicated for three-dimensional coordinate systems. Whether you choose to use Cartesian or non-Cartesian coordinates, you're going to need three variables to represent all the possible locations in a three-dimensional space, and each of the coordinates is going to come with its own unit vector. The two most common three-dimensional non-Cartesian coordinate systems are cylindrical and spherical coordinates, which you can see in

Figures 1.11 and 1.12.

In cylindrical coordinates a point P is specified by *r*, Á, *z*, where *r* (some-times called ρ) is the perpendicular distance from the *z*-axis, Á is the angle measured from the *x*-axis to the projection of *r* onto the *xy* plane, and *z* is the same as the *z* in Cartesian coordinates. Here's how you find *r*, Á, and *z* if you know *x*, *y*, and *z*:

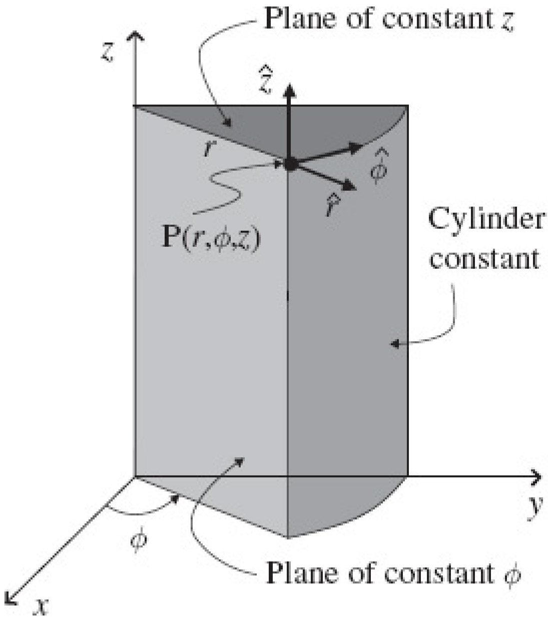


Figure 1.11 Cylindrical coordinates.

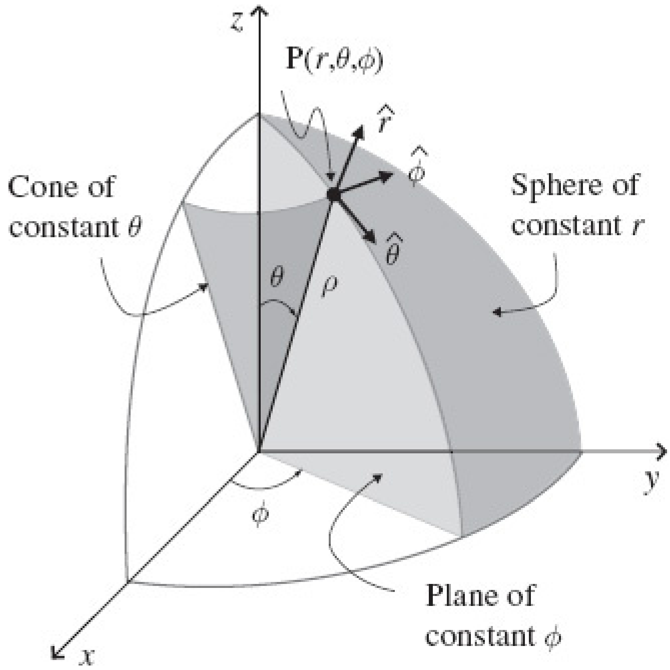


Figure 1.12 Spherical coordinates.

$$
\begin{aligned}
r &= \sqrt{x^2 + y^2} \\
\phi &= \arctan\left(\frac{y}{x}\right) \\
z &= z.
\end{aligned}
\tag{1.12}
$$

And if you have the values of *r*, Á, and *z*, you can find *x*, *y*, and *z* using

$$
\begin{aligned}
x &= r\cos(\phi) \\
y &= r\sin(\phi) \\
z &= z.
\end{aligned}
\tag{1.13}
$$

A vector at the point P is specified in cylindrical coordinates in terms of three mutually perpendicular components with unit vectors perpendicular to the cylinder of radius *r*, perpendicular to the plane through the *z*-axis at angle Á, and perpendicular to the *xy* plane at distance *z*. As in the Cartesian case, each cylindrical coordinate unit vector points in the direction in which that parameter is increasing, so $\hat{r}$ points in the direction of increasing *r*, $\hat{\phi}$ points in the direction of increasing Á, and $\hat{z}$ points in the direction of increasing *z*. The unit vectors ($\hat{r}$, $\hat{\phi}$, $\hat{z}$) form a right-handed set, so if you point the fingers of your right hand along $\hat{r}$ and push it into $\hat{\phi}$ with your right palm, your right thumb will show you the direction of $\hat{z}$.

The following equations relate the Cartesian to the cylindrical unit vectors:

$$
\begin{aligned}
\hat{r} &= \cos(\phi)\hat{\imath} + \sin(\phi)\hat{\jmath} \\
\hat{\phi} &= -\sin(\phi)\hat{\imath} + \cos(\phi)\hat{\jmath} \\
\hat{z} &= \hat{z}.
\end{aligned}
\tag{1.14}
$$

In spherical coordinates a point P is specified by *r*, θ, Á where *r* represents the distance from the origin, θ is the angle measured from the *z*-axis toward the *xy* plane, and Á is the angle measured from the *x*-axis (or *xz* plane) to the constant-Á plane containing point P. With the *z*-axis up, θ is sometimes called the zenith angle and Á the azimuth angle. You can determine the spherical coordinates *r*, θ, and Á, from *x*, *y*, and *z* using the following equations:

$$
\begin{aligned}
r &= \sqrt{x^2 + y^2 + z^2} \\
\theta &= \arccos\left(\frac{z}{\sqrt{x^2 + y^2 + z^2}}\right) \\
\phi &= \arctan\left(\frac{y}{x}\right).
\end{aligned}
\tag{1.15}
$$

And you can find *x*, *y*, and *z* from *r*, θ, and Á using:

$$
\begin{aligned}
x &= r\sin(\theta)\cos(\phi) \\
y &= r\sin(\theta)\sin(\phi) \\
z &= r\cos(\theta).
\end{aligned}
\tag{1.16}
$$

In spherical coordinates, a vector at the point P is specified in terms of three mutually perpendicular components with unit vectors perpendicular to the sphere of radius *r*, perpendicular to the plane through the *z*-axis at angle Á, and perpendicular to the cone of angle θ. The unit vectors ($\hat{r}$, $\hat{\theta}$, $\hat{\phi}$) form a right-handed set, and are related to the Cartesian unit vectors as follows:

$$
\begin{aligned}
\hat{r} &= \sin(\theta)\cos(\phi)\hat{\imath} + \sin(\theta)\sin(\phi)\hat{\jmath} + \cos(\theta)\hat{k} \\
\hat{\theta} &= \cos(\theta)\cos(\phi)\hat{\imath} + \cos(\theta)\sin(\phi)\hat{\jmath} - \sin(\theta)\hat{k} \\
\hat{\phi} &= -\sin(\phi)\hat{\imath} + \cos(\phi)\hat{\jmath}.
\end{aligned}
\tag{1.17}
$$