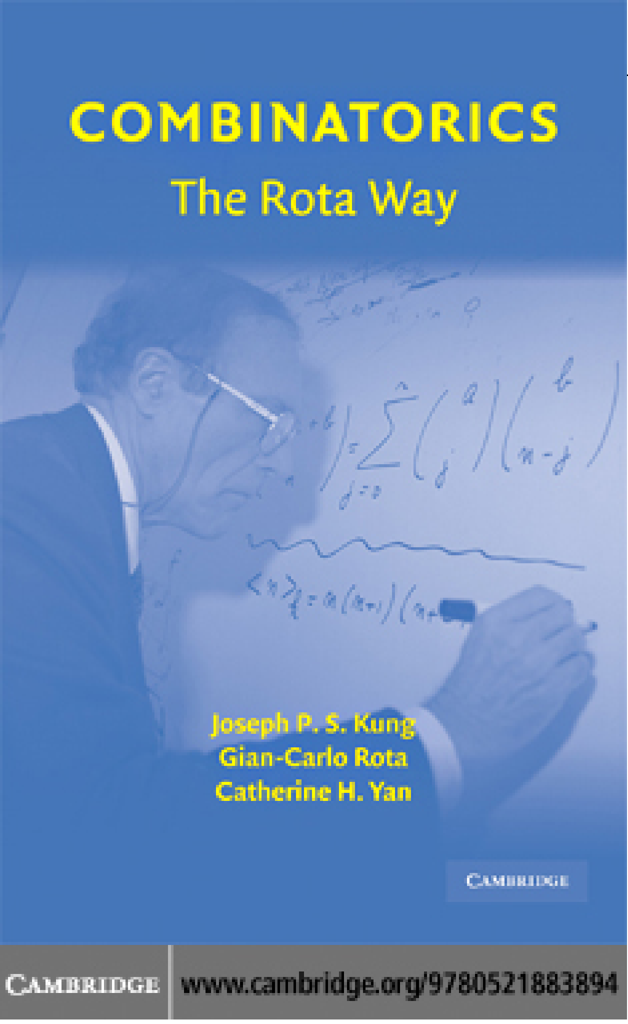
COMBINATORICS
The Rota Way
Joseph P. S. Kung
Gian-Carlo Rota
Catherine H. Yan
CAMBRIDGE
CAMBRIDGE
www.cambridge.org/9780521883894

This page intentionally left blank

Combinatorics: The Rota Way

Gian-Carlo Rota was one of the most original and colorful mathematicians of the twentieth century. His work on the foundations of combinatorics focused on revealing the algebraic structures that lie behind diverse combinatorial areas and created a new area of algebraic combinatorics. His graduate courses influenced generations of students.

Written by two of his former students, this book is based on notes from his courses and on personal discussions with him. Topics include sets and valuations, partially ordered sets, distributive lattices, partitions and entropy, matching theory, free matrices, doubly stochastic matrices, Möbius functions, chains and antichains, Sperner theory, commuting equivalence relations and linear lattices, modular and geometric lattices, valuation rings, generating functions, umbral calculus, symmetric functions, Baxter algebras, unimodality of sequences, and location of zeros of polynomials. Many exercises and research problems are included and unexplored areas of possible research are discussed.

This book should be on the shelf of all students and researchers in combinatorics and related areas.

JOSEPH P. S. KUNG is a professor of mathematics at the University of North Texas. He is currently an editor-in-chief of *Advances in Applied Mathematics*.

GIAN-CARLO ROTA (1932–1999) was a professor of applied mathematics and natural philosophy at the Massachusetts Institute of Technology. He was a member of the National Academy of Science. He was awarded the 1988 Steele Prize of the American Mathematical Society for his 1964 paper "On the Foundations of Combinatorial Theory I. Theory of Möbius Functions." He was a founding editor of *Journal of Combinatorial Theory*.

CATHERINE H. YAN is a professor of mathematics at Texas A&M University. Prior to that, she was a Courant Instructor at New York University and a Sloan Fellow.

Cambridge Mathematical Library

Cambridge University Press has a long and honorable history of publishing in mathematics and counts many classics of the mathematical literature within its list. Some of these titles have been out of print for many years now and yet the methods they espouse are still of considerable relevance today.

The Cambridge Mathematical Library will provide an inexpensive edition of these titles in a durable paperback format and at a price that will make the books attractive to individuals wishing to add them to their personal libraries. It is intended that certain volumes in the series will have forewords, written by leading experts in the subject, which will place the title in its historical and mathematical context.

Gian-Carlo Rota, Circa 1970
Pencil drawing by Eleanor Blair

Combinatorics: The Rota Way

JOSEPH P. S. KUNG
University of North Texas

GIAN-CARLO ROTA
CATHERINE H. YAN
Texas A&M University

CAMBRIDGE
UNIVERSITY PRESS

CAMBRIDGE UNIVERSITY PRESS
Cambridge, New York, Melbourne, Madrid, Cape Town, Singapore, São Paulo

Cambridge University Press
The Edinburgh Building, Cambridge CB2 8RU, UK
Published in the United States of America by Cambridge University Press, New York

www.cambridge.org
Information on this title: www.cambridge.org/9780521883894

First published in print format 2009

ISBN-13 978-0-511-50687-1 eBook (EBL)

ISBN-13 978-0-521-88389-4 hardback

ISBN-13 978-0-521-73794-4 paperback

Contents

Preface

The working title of this book was "Combinatorics 18.315." In the private language of the Massachusetts Institute of Technology, Course 18 is Mathematics, and 18.315 is the beginning graduate course in combinatorial theory. From the 1960s to the 1990s, 18.315 was taught primarily by the three permanent faculty in combinatorics, Gian-Carlo Rota, Daniel Kleitman, and Richard Stanley. Kleitman is a problem solver, with a prior career as a theoretical physicist. His way of teaching 18.315 was intuitive and humorous. With Kleitman, mathematics is fun. The experience of a Kleitman lecture can be gleaned from the transcripts of two talks.[1] Stanley's way is the opposite of Kleitman. His lectures are careful, methodical, and packed with information. He does not waste words. The experience of a Stanley lecture is captured in the two books *Enumerative Combinatorics I* and *II*, now universally known as *EC1* and *EC2*. Stanley's work is a major factor in making algebraic combinatorics a respectable flourishing mainstream area.

It is difficult to convey the experience of a Rota lecture. Rota once said that the secret to successful teaching is to reveal the material so that at the end, the idea – and there should be only one per lecture – is obvious, ready for the audience to "take home." We must confess that we have failed to pull this off in this book. The immediacy of a lecture cannot (and should not) be frozen in the textuality of a book. Instead, we have tried to convey the method behind Rota's research. Although he would object to it being stated in such stark simplistic terms, mathematical research is not about *solving* problems; it is about *finding* the right problems. One way of finding the right problems is to look for ideas common to subjects, ranging from, say, category theory to statistics. What is shared may be the implicit algebraic structures that hide behind the technicalities, in which case finding the structure is part

[1] Kleitman (1979, 2000).

of "applied universal algebra." The famous paper *Foundations I*, which revealed the role of partially ordered sets in combinatorics, is a product of this point of view. To convey Rota's thinking, which involves all of mathematics, one must go against an *idée reçue* of textbook writing: the prerequisites for this book are, in a sense, all mathematics. However, it is the ideas, not the technical details, that matter. Thus, in a different sense, there are no prerequisites to this book: we intend that a minimum of technical knowledge is needed to seriously appreciate the text of this book. Those parts where special technical knowledge is needed, usually in the exercises, can be skimmed over.

Rota taught his courses with different topics and for different audiences. The chapters in this book reflect this. Chapter 1 is about sets, functions, relations, valuations, and entropy. Chapter 2 is mostly a survey of matching theory. It provides a case study of Rota's advice to read on the history of a subject before tackling its problems. The aim of Chapter 2 is to find what results one should expect when one extends matching theory to higher dimensions. Possible paths are suggested in Section 2.8. The third chapter offers a mixture of topics in partially ordered sets. The first section is about Möbius functions. After the mid-1960s, Möbius functions were never the focus of a Rota course; his feeling was that he had made his contribution. However, a book on Rota's combinatorics would be incomplete without Möbius functions. Other topics in Chapter 3 are Dilworth's chain partition theorem; Sperner theory; modular, linear, and geometric lattices; and valuation rings. Linear lattices, or lattices represented by commuting equivalence relations, lie at the intersection of geometric invariant theory and the foundations of probability theory. Chapter 4 is about generating functions, polynomial sequences of binomial type, and the umbral calculus. These subjects have been intensively studied and the chapter merely opens the door to this area. Chapter 5 is about symmetric functions. We define them by distribution and occupancy and apply them to the study of Baxter algebras. This chapter ends with a section on symmetric functions over finite fields. The sixth chapter is on polynomials and their zeros. The topic is motivated, in part, by unimodality conjectures in combinatorics and was the last topic Rota taught regularly. Sadly, we did not have the opportunity to discuss this topic in detail with him.

There is a comprehensive bibliography. Items in the bibliography are referenced in the text by author and year of publication. In a few cases when two items by the same authors are published in the same year, suffixes *a* and *b* are appended according to the order in which the items are listed. Exceptions are several papers by Rota and the two volumes of his selected papers; these are referenced by short titles. Our convention is explained in the beginning of the bibliography.

We should now explain the authorship and the title of this book. Gian-Carlo Rota passed away unexpectedly in 1999, a week before his 67th birthday. This book was physically written by the two authors signing this preface. We will refer to the third author simply as Rota. As for the title, we wanted one that is not boring. The word "way" is not meant to be prescriptive, in the sense of "my way or the highway." Rather, it comes from the core of the cultures of the three authors. The word "way" resonates with the word "cammin" in the first line of Dante's *Divina commedia*, "Nel mezzo del cammin di nostra vita." It also resonates as the character "tao" in Chinese. In both senses, the way has to be struggled for and sought individually. This is best expressed in Chinese:

道可道，非常道

Inadequately translated into rectilinear English, this says "a way which can be *wayed* (that is, taught or followed) cannot be a way." Rota's way is but one way of doing combinatorics. After "seeing through" Rota's way, the reader will seek his or her own way.

It is our duty and pleasure to thank the many friends who have contributed, knowingly or unknowingly, to the writing of this book. There are several sets of notes from Rota's courses. We have specifically made use of our own notes (1976, 1977, 1994, and 1995), and more crucially, our recollection of many conversations we had with Rota. Norton Starr provided us with his notes from 1964. These offer a useful pre-foundations perspective. We have also consulted notes by Miklós Bona, Gabor Hetyei, Richard Ehrenborg, Matteo Mainetti, Brian Taylor, and Lizhao Zhang from the early 1990s. We have benefited from discussions with Ottavio D'Antona, Wendy Chan, and Dan Klain. John Guidi generously provided us with his verbatim transcript from 1998, the last time Rota taught 18.315. Section 1.4 is based partly on notes of Kenneth Baclawski, Sara Billey, Graham Sterling, and Carlo Mereghetti. Section 2.8 originated in discussions with Jay Sulzberger in the 1970s. Sections 3.4 and 3.5 were much improved by a discussion with J. B. Nation. William Y. C. Chen and his students at the Center for Combinatorics at Nankai University (Tianjin, China) – Thomas Britz, Dimitrije Kostic, Svetlana Poznanovik, and Susan Y. Wu – carefully read various sections of this book and saved us from innumerable errors. We also thank Ester Rota Gasperoni, Gian-Carlo's sister, for her encouragement of this project.

Finally, Joseph Kung was supported by a University of North Texas faculty development leave. Catherine Yan was supported by the National Science Foundation and a faculty development leave funded by the Association of Former Students at Texas A&M University.

May 2008

Joseph P. S. Kung
Catherine H. Yan

1

Sets, Functions, and Relations

1.1 Sets, Valuations, and Boolean Algebras

We shall usually work with finite sets. If A is a finite set, let $|A|$ be the number of elements in A. The function $|\cdot|$ satisfies the functional equation

$$|A \cup B| + |A \cap B| = |A| + |B|.$$

The function $|\cdot|$ is one of many functions measuring the "size" of a set. Let v be a function from a collection $\mathcal{C}$ of sets to an algebraic structure $\mathbb{A}$ (such as an Abelian group or the nonnegative real numbers) on which a commutative binary operation analogous to addition is defined. Then v is a *valuation* if for sets A and B in $\mathcal{C}$,

$$v(A \cup B) + v(A \cap B) = v(A) + v(B),$$

whenever the union $A \cup B$ and the intersection $A \cap B$ are in $\mathcal{C}$.

Sets can be combined algebraically and sometimes two sets can be compared with each other. The operations of union $\cup$ and intersection $\cap$ are two basic algebraic binary operations on sets. In addition, if we fix a universal set S containing all the sets we will consider, then we have the unary operation A^c of *complementation,* defined by

$$A^c = S \backslash A = \{a\colon a \in S \text{ and } a \notin A\}.$$

Sets are partially ordered by containment. A collection $\mathcal{C}$ of subsets is a *ring of sets* if $\mathcal{C}$ is closed under unions and intersections. If, in addition, all the sets in $\mathcal{C}$ are subsets of a universal set and $\mathcal{C}$ is closed under complementation, then $\mathcal{C}$ is a *field of sets*. The collection 2^S of all subsets of the set S is a field of sets.

Boolean algebras capture the algebraic and order structure of fields of sets. The axioms of a Boolean algebra abstract the properties of union, intersection, and complementation, without any mention of elements or points. As John von Neumann put it, the theory of Boolean algebras is "pointless" set theory.

A *Boolean algebra* P is a set with two binary operations, the *join* $\vee$ and the *meet* $\wedge$; a unary operation, *complementation* $\cdot^c$; and two nullary operations or constants, the *minimum* $\hat{0}$ and the *maximum* $\hat{1}$. The binary operations $\vee$ and $\wedge$ satisfy the *lattice axioms*:

L1. Idempotency: $x \vee x = x, \quad x \wedge x = x.$
L2. Commutativity: $x \vee y = y \vee x, \quad x \wedge y = y \wedge x.$
L3. Associativity: $x \vee (y \vee z) = (x \vee y) \vee z, \quad x \wedge (y \wedge z) = (x \wedge y) \wedge z.$
L4. Absorption: $x \wedge (x \vee y) = x, \quad x \vee (x \wedge y) = x.$

Joins and meets also satisfy the *distributive axioms*

$$x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z), \quad x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z).$$

In addition, the five operations satisfy the *De Morgan laws*

$$(x \vee y)^c = x^c \wedge y^c, \quad (x \wedge y)^c = x^c \vee y^c,$$

two pairs of rules concerning complementation

$$x \vee x^c = \hat{1}, \quad x \wedge x^c = \hat{0}$$

and

$$\hat{0} \neq \hat{1}, \ \ \hat{1}^c = \hat{0}, \ \ \hat{0}^c = \hat{1}.$$

It follows from the axioms that complementation is an involution; that is, $(x^c)^c = x$. The smallest Boolean algebra is the algebra $\underline{2}$ with two elements $\hat{0}$ and $\hat{1}$, thought of as the *truth values* "false" and "true." The axioms are, more or less, those given by George Boole. Boole, perhaps the greatest simplifier in history, called these axioms "the laws of thought."[1] He may be right, at least for silicon-based intelligence.

A *lattice* is a set L with two binary operations $\vee$ and $\wedge$ satisfying axioms L1–L4. A *partially ordered set* or *poset* is a set P with a relation $\leq$ (or $\leq_P$ when we need to be clear which partial order is under discussion) satisfying three axioms:

PO1. Reflexivity: $x \leq x$.
PO2. Transitivity: $x \leq y$ and $y \leq z$ imply $x \leq z$.
PO3. Antisymmetry: $x \leq y$ and $y \leq x$ imply $x = y$.

[1] Boole (1854). For careful historical studies, see, for example, Hailperin (1986) and Smith (1982).

The *order-dual* $P^{\downarrow}$ is the partial order obtained from P by inverting the order; that is,

$$x \leq_{P^{\downarrow}} y \quad \text{if and only if} \quad y \leq_P x.$$

Sets are partially ordered by containment. This order relation is not explicit in a Boolean algebra, but can be defined by using the meet or the join. More generally, in a lattice L, we can define a partial order $\leq_L$ compatible with the lattice operations on L by $x \leq_L y$ if and only if $x \wedge y = x$. Using the absorption axiom L4, it is easy to prove that $x \wedge y = x$ if and only if $x \vee y = y$; thus, the following three conditions are equivalent:

$$x \leq_L y, \quad x \wedge y = x, \quad x \vee y = y.$$

The join $x \vee y$ is the *supremum* or *least upper bound* of x and y in the partial order $\leq_L$; that is, $x \vee y \geq_L x$, $x \vee y \geq_L y$, and if $z \geq_L x$ and $z \geq_L y$, then $z \geq_L x \vee y$. The meet $x \wedge y$ is the *infimum* or *greatest lower bound* of x and y. Supremums and infimums can be defined for arbitrary sets in partial orders, but they need not exist, even when the partial order is defined from a lattice. However, supremums and infimums of finite sets always exist in lattices.

By the De Morgan laws, the complementation map $x \mapsto x^c$ from a Boolean algebra P to itself exchanges the operations $\vee$ and $\wedge$. This gives an (order) duality: if a statement P about Boolean algebra holds for all Boolean algebras, then the statement $P^{\downarrow}$, obtained from P by the exchanges $x \leftrightarrow x^c$, $\wedge \leftrightarrow \vee$, $\leq \leftrightarrow \geq$, $\hat{0} \leftrightarrow \hat{1}$, is also valid over all Boolean algebras. A similar duality principle holds for statements about lattices.

We end this section with representation theorems for Boolean algebras as fields of subsets. Let P and Q be Boolean algebras. A function $\phi : P \to Q$ is a *Boolean homomorphism* or *morphism* if

$$\begin{aligned} \phi(x \vee y) &= \phi(x) \vee \phi(y) \\ \phi(x \wedge y) &= \phi(x) \wedge \phi(y) \\ \phi(x^c) &= (\phi(x))^c. \end{aligned}$$

1.1.1. Theorem. A finite Boolean algebra P is isomorphic to the Boolean algebra 2^S of all subsets of a finite set S.

Proof. An *atom* a in P is an element covering the minimum $\hat{0}$; that is, $a > \hat{0}$ and if $a \geq b > \hat{0}$, then $b = a$. Atoms correspond to one-element subsets. Let S be the set of atoms of B and $\psi : P \to 2^S$, $\phi : 2^S \to P$ be the functions defined by

$$\psi(x) = \{a\colon a \in S, a \leq x\}, \quad \phi(A) = \bigvee_{a \in A} a.$$

It is routine to check that both compositions $\psi\phi$ and $\phi\psi$ are identity functions and that ϕ and ψ are Boolean morphisms. $\square$

The theorem is false if finiteness is not assumed. Two properties implied by finiteness are needed in the proof. A Boolean algebra P is *complete* if the supremum and infimum (with respect to the partial order $\leq$ defined by the lattice operations) exist for every subset (of any cardinality) of elements in P. It is *atomic* if every element x in P is a supremum of atoms. The proof of Theorem 1.1.1 yields the following result.

1.1.2. Theorem. A Boolean algebra P is isomorphic to a Boolean algebra 2^S of *all* subsets of a set if and only if P is complete and atomic.

Theorem 1.1.2 says that not all Boolean algebras are of the form 2^S for some set S. For a specific example, let S be an infinite set. A subset in S is *cofinite* if its complement is finite. The *finite–cofinite Boolean algebra* on the set S is the Boolean algebra formed by the collection of all finite or cofinite subsets of S. The finite–cofinite algebra on an infinite set is atomic but not complete. Another example comes from analysis. The algebra of measurable sets of the real line, modulo the sets of measure zero, is a nonatomic Boolean algebra in which unions and intersections of countable families of equivalence classes of sets exist.

One might hope to represent a Boolean algebra as a field of subsets constructed from a topological space. The collection of open sets is a natural choice. However, because complements exist and complements of open sets are closed, we need to consider *clopen* sets, that is, sets that are both closed and open.

1.1.3. Lemma. The collection of clopen sets of a topological space is a field of subsets (and forms a Boolean algebra).

Since meets and joins are finitary operations, it is natural to require the topological space to be compact. A space X is *totally disconnected* if the only connected subspaces in X are single points. If we assume that X is compact and Hausdorff, then being totally connected is equivalent to each of the two conditions: (a) every open set is the union of clopen sets, or (b) if p and q are two points in X, then there exists a clopen set containing p but not q. A *Stone space* is a totally disconnected compact Hausdorff space.

1.1.4. The Stone representation theorem.[2] Every Boolean algebra can be represented as the field of clopen sets of a Stone space.

[2] Stone (1936).

There are two ways, topological or algebraic, to prove the Stone representation theorem. In both, the key step is to construct a Stone space X from a Boolean algebra P. A $\underline{2}$-*morphism* of P is a Boolean morphism from P onto the two-element Boolean algebra $\underline{2}$. Let X be the set of $\underline{2}$-morphisms of P. Regarding X as a (closed) subset of the space $\underline{2}^P$ of all functions from P into $\underline{2}$ with the product topology, we obtain a Stone space. Each element x in P defines a continuous function $X \to \underline{2}$, $f \mapsto f(x)$. Using this, we obtain a Boolean morphism from P into the Boolean algebra of clopen sets of X.

The algebraic approach regards a Boolean algebra P as a commutative ring, with addition defined by $x + y = (x \wedge y^c) \vee (x^c \wedge y)$ and multiplication defined by $xy = x \wedge y$. (Addition is an abstract version of symmetric difference of subsets.) Then the set of prime ideals $\mathrm{Spec}(P)$ of P is a topological space under the *Zariski topology*: the closed sets are the order filters in $\mathrm{Spec}(P)$ under set-containment. The order filters are also open, and hence clopen. Then the Boolean algebra P is isomorphic to the Boolean algebra of clopen sets of $\mathrm{Spec}(P)$. Note that in a ring constructed from a Boolean algebra, $2x = x + x = 0$ for all x. In such a ring, every prime ideal is maximal. Maximal ideals are in bijection with $\underline{2}$-morphisms and so $\mathrm{Spec}(P)$ and X are the same set (and less obviously, the same topological space).[3]

The Boolean operations on a field P of subsets of a universal set S can be modeled by addition and multiplication over a ring $\mathbb{A}$ using indicator (or characteristic) functions. If S is a universal set and $A \subseteq S$, then the *indicator function* χ_A *of* A is the function $S \to \mathbb{A}$ defined by

$$\chi_A(a) = \begin{cases} 1 & \text{if } a \in A, \\ 0 & \text{if } a \notin A. \end{cases}$$

The indicator function satisfies

$$\begin{aligned} \chi_{A\cap B}(a) &= \chi_A(a)\chi_B(a), \\ \chi_{A\cup B}(a) &= \chi_A(a) + \chi_B(a) - \chi_A(a)\chi_B(a). \end{aligned}$$

When $\mathbb{A}$ is GF(2), the (algebraic) field of integers modulo 2, then the indicator function gives an injection from P to the vector space $\mathrm{GF}(2)^S$ of dimension $|S|$ with coordinates labeled by S. Since GF(2) is the Boolean algebra $\underline{2}$ as a ring, indicator functions also give an injection into the Boolean algebra $\underline{2}^{|S|}$. Indicator functions give another way to prove Theorem 1.1.1.

It will be useful to have the notion of a multiset. Informally, a multiset is a set in which elements can occur in multiple copies. For example, $\{a, a, b, a, b, c\}$

[3] See Halmos (1974) for the topological approach. A no-nonsense account of the algebraic approach is in Atiyah and MacDonald (1969, p. 14). See also Johnstone (1982).

is a multiset in which the element a occurs with multiplicity 3. One way to define multisets formally is to generalize indicator functions. If S is a universal set and $A \subseteq S$, then a *multiset* M is defined by a *multiplicity function* $\chi_M : S \to \mathbb{N}$ (where $\mathbb{N}$ is the set of nonnegative integers). The *support* of M is the subset $\{a \in S\colon \chi_M(a) > 0\}$. Unions and intersections of multisets are defined by

$$\chi_{A\cap B}(a) = \min\{\chi_A(a), \chi_B(a)\},$$
$$\chi_{A\cup B}(a) = \max\{\chi_A(a), \chi_B(a)\}.$$

We have defined union so that it coincides with set-union when both multisets are sets. We also have the notion of the *sum* of two multisets, defined by

$$\chi_{A+B}(a) = \chi_A(a) + \chi_B(a).$$

This sum is an analog of disjoint union for sets.

Exercises

1.1.1. *Distributive and shearing inequalities.*

Let L be a lattice. Prove that for all $x, y, z \in L$,

$$(x \wedge y) \vee (x \wedge z) \leq x \wedge (y \vee z)$$

and

$$(x \wedge y) \vee (x \wedge z) \leq x \wedge (y \vee (x \wedge z)).$$

1.1.2. *Sublattices forbidden by the distributive axioms.*[4]

A *sublattice* of a lattice L is a subset of elements of L closed under meets and joins. Show that a lattice L is distributive if and only if L does not contain the *diamond* M_5 and the *pentagon* N_5 as a sublattice (see Figure 1.1).

1.1.3. *More on the distributive axioms.*

(a) Assuming the lattice axioms, show that the two identities in the distributive axioms imply each other. Show that each identity is equivalent to the self-dual identity

$$(x \vee y) \wedge (y \vee z) \wedge (z \vee x) = (x \wedge y) \vee (y \wedge z) \vee (z \wedge x).$$

(b) Show that a lattice L is distributive if and only if for all $a, x, y \in L$, $a \vee x = a \vee y$ and $a \wedge x = a \wedge y$ imply $x = y$.

[4] Birkhoff (1934).

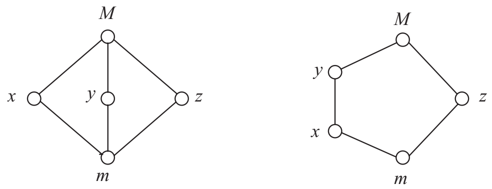

Figure 1.1 The diamond and the pentagon.

1.1.4. *Implication.*

Define the binary operation $\to$ of *implication* on a Boolean algebra P by

$$x \to y = x^c \vee y.$$

Show that the binary operation $\to$ and the constant $\hat{0}$ generate the operations $\vee, \wedge, \cdot^c$ and the constant $\hat{1}$. Give a set of axioms using $\to$ and $\hat{0}$.

1.1.5. *Conditional disjunction.*

Define the ternary operation $[x, y, z]$ of *conditional disjunction* by

$$[x, y, z] = (x \wedge y) \vee (y \wedge z) \vee (z \wedge x).$$

Note that $[x, y, z]$ is invariant under permutations of the variables. Show that $\vee$ and $\wedge$ can be defined using conditional disjunction and the constants $\hat{1}$ and $\hat{0}$. Find an elegant set of axioms for Boolean algebras using conditional disjunction and complementation.

1.1.6. *Huntington's axiom.*[5]

Show that a Boolean algebra P can be defined as a nonempty set with a binary operation $\vee$ and a unary operation $\cdot^c$ satisfying the following three axioms:

H1. $\vee$ is associative.
H2. $\vee$ is commutative.
H3. *Huntington's axiom:* For all x and y,

$$(x^c \vee y^c)^c \vee (x^c \vee y)^c = x.$$

1.1.7. *The Sheffer stroke.*[6]

Show that a Boolean algebra P can be defined as a set P with at least two elements with single binary operation $|$ satisfying the axioms:

Sh1. $(a|a)|(a|a) = a$.

[5] Huntington (1933). [6] Sheffer (1913).

Sh2. $a|(b|(b|b)) = a|a$.
Sh3. $(a|(b|c))|(a|(b|c)) = ((b|b)|a)|((c|c)|a)$.

1.1.8. Let S be the countable set $\{1/n\colon 1 \le n < \infty\}$ and consider the topological space $S \cup \{0\}$ with the topology induced from the real numbers. Show that the finite–cofinite algebra on S is the collection of open sets of $S \cup \{0\}$.

1.1.9. (a) Let $\mathcal{H}$ be the collection of all unions of a finite number of subsets of rational numbers of the following form:

$$\{r\colon r < b\},\ \{r\colon a \le r < b\},\ \text{ or } \{r\colon a \le r\}.$$

Show that $\mathcal{H}$ is a countable Boolean algebra (under set-containment) with no atoms.

(b) Show that any two countable Boolean algebras with no atoms are isomorphic.

1.1.10. Is there a natural description of the Stone space of the Boolean algebra of measurable sets of real numbers modulo sets of measure zero?

1.1.11. *Infinite distributive axioms.*

The infinite distributive axioms for the lattice operations say

$$\bigwedge_{i:\,i\in I}\ \bigvee_{j:\,j\in J} x_{ij} = \bigvee_{f:\,f:I\to J}\ \bigwedge_{i:\,i\in I} x_{i,f(i)}, \qquad \bigvee_{i:\,i\in I}\ \bigwedge_{j:\,j\in J} x_{ij} = \bigwedge_{f:\,f:I\to J}\ \bigvee_{i:\,i\in I} x_{i,f(i)},$$

with f ranging over all functions from I to J. To see that this is the correct infinite extension, interpret $\wedge$ as multiplication and $\vee$ as addition. Then formally

$$(x_{11} + x_{12} + x_{13} + \cdots)(x_{21} + x_{22} + x_{23} + \cdots)(x_{31} + x_{32} + x_{33} + \cdots)\cdots$$
$$= \sum_{f:\,f:I\to J} x_{1,f(1)}x_{2,f(2)}x_{3,f(3)}\cdots.$$

Prove the following theorem of Tarski.[7] Let P be a Boolean algebra. Then the following conditions are equivalent:

1. P is complete and satisfies the infinite distributivity axioms.
2. P is complete and atomic.
3. P is the Boolean algebra of all subsets of a set.

1.1.12. *Universal valuations for finite sets.*

Let S be a finite set, $\{x_a\colon a \in S\}$ be a set of variables, one for each element of S, x_0 be another variable, and $\mathbb{A}[\underline{x}]$ be the ring of polynomials in the

[7] Tarski (1929).

set of variables $\{x_a : a \in S\} \cup \{x_0\}$ with coefficients in a ring $\mathbb{A}$. Show that $v : 2^S \to \mathbb{A}[\underline{x}]$ defined by

$$v(A) = x_0 + \sum_{a : a \in A} x_a$$

is a valuation taking values in $\mathbb{A}[\underline{x}]$ and every valuation taking values in $\mathbb{A}$ can be obtained by assigning a value in $\mathbb{A}$ to each variable in $\{x_a : a \in S\} \cup \{x_0\}$.

1.2 Partially Ordered Sets

Let P be a partially ordered set. An element x *covers* the element y in the partially ordered set if $x > y$ and there is no element z in P such that $x > z > y$. An element m is *minimal* in the partial order P if there are no elements y in P such that $y < m$. A *maximal* element is a minimal element in the dual $P^{\downarrow}$.

Two elements x and y in P are *comparable* if $x \leq y$ or $y \leq x$; they are *incomparable* if neither $x \leq y$ nor $y \leq x$. A subset $C \subseteq P$ is a *chain* if any two elements in C are comparable. A subset $A \subseteq P$ is an *antichain* if any two elements in A are incomparable. If C is a finite chain and $|C| = n + 1$, then the elements in C can be *linearly ordered,* so that

$$x_0 < x_1 < x_2 < \cdots < x_n.$$

The *length* of the chain C is n, 1 less than the number of elements in C. A chain $x_0 < x_1 < \cdots < x_n$ in the partial order P is *maximal* or *saturated* if x_{i+1} covers x_i for $1 \leq i \leq n$. A function r defined from P to the nonnegative integers is a *rank function* if $r(x) = 0$ for every minimal element and $r(y) = r(x) + 1$ whenever y covers x. The partial order P is *ranked* if there exists a rank function on P. The *rank* of the entire partially ordered set P is the maximum $\max\{r(x) : x \in P\}$. If $x \leq y$ in P, the *interval* $[x, y]$ is the set $\{z : x \leq z \leq y\}$.

If P is finite, then we draw a picture of P by assigning a vertex or dot to each element of P and putting a directed edge or arrow from y to x if x covers y. Thinking of the arrows as flexible, we can draw the picture so that if $x > y$, then x is above y. It is not required that the edges do not cross each other. Helmut Hasse drew such pictures for field extensions. For this reason, pictures of partial orders are often called *Hasse diagrams.*

Let P and Q be partially ordered sets. A function $f : P \to Q$ is *order-preserving* if for elements x and y in P, $x \leq_P y$ implies $f(x) \leq_Q f(y)$. A function f is *order-reversing* if $x \leq_P y$ implies $f(x) \geq_Q f(y)$.

A subset $I \subseteq P$ is an *(order) ideal* of P if it is "down-closed;" that is, $y \leq x$ and $x \in I$ imply $y \in I$. Note that we do not require ideals to be closed under joins if P is a lattice. The union and intersection of an arbitrary collection of ideals are ideals. There is a bijection between ideals and antichains: an ideal I is associated with the antichain $A(I)$ of maximal elements in I. If a is an element of P, then the set $I(a)$ defined by

$$I(a) = \{x\colon x \le a\}$$

is an ideal. An ideal is *principal* if it has this form or, equivalently, if it has exactly one maximal element a. The element a *generates* the principal ideal $I(a)$.

If A is a set of elements of P, then the *ideal* $I(A)$ *generated by* A is the ideal defined, in two equivalent ways, by

$$I(A) = \{x\colon x \le a \text{ for some } a \in A\}$$

or

$$I(A) = \bigcup_{a\colon a \in A} I(a).$$

Ideals are also in bijection with order-preserving functions from P to the Boolean algebra $\underline{2}$: the ideal I corresponds to the function $f : P \to \underline{2}$ defined by $f(x) = \hat{0}$ if $x \in I$ and $f(x) = \hat{1}$ otherwise.

Filters are "up-closed;" in other words, filters are ideals in the order-dual $P^{\downarrow}$. The set complement $P \backslash I$ of an ideal is a filter. Any statement about ideals inverts to a statement about filters. In particular, the map sending a filter to the antichain of its minimal elements is a bijection. Hence, there is a bijection between the ideals and the filters of a partially ordered set. If A is a set of elements of P, then the *filter* $F(A)$ *generated by* A is the filter defined by

$$F(A) = \{x\colon x \ge a \text{ for some } a \in A\}.$$

When A is a single-element set $\{a\}$, the filter $F(\{a\})$, written $F(a)$, is the *principal filter* generated by a.

Let P be a partial order and Q be a partial order on the same set P. The partial order Q is an *extension* of P if $x \le_P y$ implies $x \le_Q y$ or, equivalently, as a subset of the Cartesian product $P \times P$, the relation $\le_P$ is contained in $\le_Q$. If Q is a chain, then it is a *linear extension* of P.

1.2.1. Lemma.[8] Let P be a finite partially ordered set. If x is incomparable with y, then there is a linear extension L of P such that $x <_L y$.

Proof. We can construct a linear extension in the following way: let $\min(P)$ be the set of minimal elements of P. Then choose an element x_1 from $\min(P)$,

[8] Dushnik and Miller (1941).