

Springer Undergraduate Mathematics Series

S

U

M

S

Luis Barreira
Claudia Valls

Complex Analysis and Differential Equations

 Springer

Springer Undergraduate Mathematics Series

Advisory Board

M.A.J. Chaplain *University of Dundee, Dundee, Scotland, UK*

K. Erdmann *University of Oxford, Oxford, England, UK*

A. MacIntyre *Queen Mary, University of London, London, England, UK*

E. Süli *University of Oxford, Oxford, England, UK*

M.R. Tehranchi *University of Cambridge, Cambridge, England, UK*

J.F. Toland *University of Bath, Bath, England, UK*

For further volumes:

www.springer.com/series/3423

Luis Barreira • Claudia Valls

Complex Analysis
and Differential
Equations

 Springer

Luis Barreira
Departamento de Matemática
Instituto Superior Técnico
Lisboa, Portugal

Claudia Valls
Departamento de Matemática
Instituto Superior Técnico
Lisboa, Portugal

Based on translations from the Portuguese language editions:

Análise Complexa e Equações Diferenciais by Luis Barreira
Copyright © IST Press 2009, Instituto Superior Técnico

Exercícios de Análise Complexa e Equações Diferenciais by Luís Barreira and Cláudia Valls
Copyright © IST Press 2010, Instituto Superior Técnico
All Rights Reserved

ISSN 1615-2085 Springer Undergraduate Mathematics Series
ISBN 978-1-4471-4007-8 ISBN 978-1-4471-4008-5 (eBook)
DOI 10.1007/978-1-4471-4008-5
Springer London Heidelberg New York Dordrecht

Library of Congress Control Number: 2012936978

Mathematics Subject Classification: 30-01, 34-01, 35-01, 42-01

© Springer-Verlag London 2012

This work is subject to copyright. All rights are reserved by the Publisher, whether the whole or part of the material is concerned, specifically the rights of reprinting, reuse of illustrations, recitation, broadcasting, reproduction on microfilms or in any other physical way, and transmission or information storage and retrieval, electronic adaptation, computer software, or by similar or dissimilar methodology now known or hereafter developed. Exempted from this legal reservation are brief excerpts in connection with reviews or scholarly analysis or material supplied specifically for the purpose of being entered and executed on a computer system, for exclusive use by the purchaser of the work. Duplication of this publication or parts thereof is permitted only under the provisions of the Copyright Law of the Publisher's location, in its current version, and permission for use must always be obtained from Springer. Permissions for use may be obtained through RightsLink at the Copyright Clearance Center. Violations are liable to prosecution under the respective Copyright Law.

The use of general descriptive names, registered names, trademarks, service marks, etc. in this publication does not imply, even in the absence of a specific statement, that such names are exempt from the relevant protective laws and regulations and therefore free for general use.

While the advice and information in this book are believed to be true and accurate at the date of publication, neither the authors nor the editors nor the publisher can accept any legal responsibility for any errors or omissions that may be made. The publisher makes no warranty, express or implied, with respect to the material contained herein.

Printed on acid-free paper

Springer is part of Springer Science+Business Media (www.springer.com)

Preface

This book is essentially two books in one. Namely, it is an introduction to two large areas of mathematics—*complex analysis* and *differential equations*—and the material is naturally divided into two parts. This includes holomorphic functions, analytic functions, ordinary differential equations, Fourier series, and partial differential equations. Moreover, half of the book consists of approximately 200 worked-out problems plus 200 exercises of variable level of difficulty. The worked-out problems fill the gap between the theory and the exercises.

To a considerable extent, the parts of complex analysis and differential equations can be read independently. In the second part, some special emphasis is given to the applications of complex analysis to differential equations. On the other hand, the material is still developed with sufficient detail in order that the book contains an ample introduction to differential equations, and not strictly related to complex analysis.

The text is tailored to any course giving a first introduction to complex analysis or to differential equations, assuming as prerequisite only a basic knowledge of linear algebra and of differential and integral calculus. But it can also be used for independent study. In particular, the book contains a large number of examples illustrating the new concepts and results. Moreover, the worked-out problems, carefully prepared for each part of the theory, make this the ideal book for independent study, allowing the student to actually see how the theory applies, before solving the exercises.

Lisbon, Portugal

Luis Barreira and Claudia Valls

Contents

Part I Complex Analysis

| | | |
|----------|---|-----|
| 1 | Basic Notions | 3 |
| 1.1 | Complex Numbers | 3 |
| 1.2 | Polar Form | 8 |
| 1.3 | Conjugate | 11 |
| 1.4 | Complex Functions | 14 |
| 1.5 | Solved Problems and Exercises | 21 |
| 2 | Holomorphic Functions | 37 |
| 2.1 | Limits and Continuity | 37 |
| 2.2 | Differentiability | 39 |
| 2.3 | Differentiability Condition | 48 |
| 2.4 | Paths and Integrals | 52 |
| 2.5 | Primitives | 62 |
| 2.6 | Index of a Closed Path | 69 |
| 2.7 | Cauchy's Integral Formula | 73 |
| 2.8 | Integrals and Homotopy of Paths | 74 |
| 2.9 | Harmonic Conjugate Functions | 78 |
| 2.10 | Solved Problems and Exercises | 82 |
| 3 | Sequences and Series | 109 |
| 3.1 | Sequences | 109 |
| 3.2 | Series of Complex Numbers | 111 |

| | | |
|---------------------------------------|--|------------|
| 3.3 | Series of Real Numbers | 115 |
| 3.4 | Uniform Convergence | 122 |
| 3.5 | Solved Problems and Exercises | 128 |
| 4 | Analytic Functions | 149 |
| 4.1 | Power Series | 149 |
| 4.2 | Zeros | 162 |
| 4.3 | Laurent Series and Singularities | 164 |
| 4.4 | Residues | 174 |
| 4.5 | Meromorphic Functions | 177 |
| 4.6 | Solved Problems and Exercises | 183 |
| Part II Differential Equations | | |
| 5 | Ordinary Differential Equations | 223 |
| 5.1 | Basic Notions | 223 |
| 5.2 | Existence and Uniqueness of Solutions | 226 |
| 5.3 | Linear Equations: Scalar Case | 233 |
| 5.4 | Linear Equations: General Case | 236 |
| 5.5 | Computing Exponentials of Matrices | 244 |
| 5.6 | Solved Problems and Exercises | 250 |
| 6 | Solving Differential Equations | 281 |
| 6.1 | Exact Equations | 281 |
| 6.2 | Equations Reducible to Exact | 285 |
| 6.3 | Scalar Equations of Order Greater than 1 | 287 |
| 6.4 | Laplace Transform | 296 |
| 6.5 | Solved Problems and Exercises | 311 |
| 7 | Fourier Series | 333 |
| 7.1 | An Example | 333 |
| 7.2 | Fourier Series | 338 |
| 7.3 | Uniqueness and Orthogonality | 347 |
| 7.4 | Even and Odd Functions | 353 |
| 7.5 | Series of Cosines and Series of Sines | 355 |
| 7.6 | Integration and Differentiation Term by Term | 359 |
| 7.7 | Solved Problems and Exercises | 362 |
| 8 | Partial Differential Equations | 373 |
| 8.1 | Heat Equation and Its Modifications | 373 |
| 8.2 | Laplace Equation | 383 |
| 8.3 | Wave Equation | 388 |
| 8.4 | Solved Problems and Exercises | 392 |
| Index | | 413 |

Part I
Complex Analysis

1

Basic Notions

In this chapter we introduce the set of complex numbers, as well as some basic notions. In particular, we describe the operations of addition and multiplication, as well as the powers and roots of complex numbers. We also introduce various complex functions that are natural extensions of corresponding functions in the real case, such as the exponential, the cosine, the sine, and the logarithm.

1.1 Complex Numbers

We first introduce the set of complex numbers as the set of pairs of real numbers equipped with operations of addition and multiplication.

Definition 1.1

The set \mathbb{C} of *complex numbers* is the set \mathbb{R}^2 of pairs of real numbers equipped with the operations

$$(a, b) + (c, d) = (a + c, b + d) \quad (1.1)$$

and

$$(a, b) \cdot (c, d) = (ac - bd, ad + bc) \quad (1.2)$$

for each $(a, b), (c, d) \in \mathbb{R}^2$.

One can easily verify that the operations of addition and multiplication in (1.1) and (1.2) are commutative, that is,

$$(a, b) + (c, d) = (c, d) + (a, b)$$

and

$$(a, b) \cdot (c, d) = (c, d) \cdot (a, b)$$

for every $(a, b), (c, d) \in \mathbb{R}^2$.

Example 1.2

For example, we have

$$(5, 4) + (3, 2) = (8, 6)$$

and

$$(2, 1) \cdot (-1, 6) = (2 \cdot (-1) - 1 \cdot 6, 2 \cdot 6 + 1 \cdot (-1)) = (-8, 11).$$

For simplicity of notation, we always write

$$(a, 0) = a,$$

thus identifying the pair $(a, 0) \in \mathbb{R}^2$ with the real number a (see Figure 1.1). We define the *imaginary unit* by

$$(0, 1) = i$$

(see Figure 1.1).

Proposition 1.3

We have $i^2 = -1$ and $a + ib = (a, b)$ for every $a, b \in \mathbb{R}$.

Proof

Indeed,

$$i^2 = (0, 1) \cdot (0, 1) = (-1, 0) = -1,$$

and

$$\begin{aligned} a + ib &= (a, 0) + (0, 1) \cdot (b, 0) \\ &= (a, 0) + (0, b) = (a, b), \end{aligned}$$

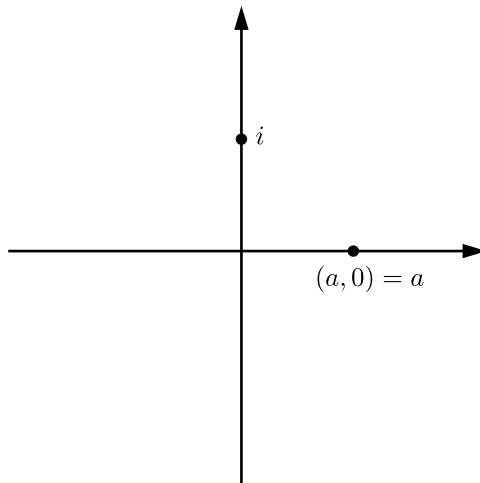


Figure 1.1 Real number a and imaginary unit i

which yields the desired statement. \square

We thus have

$$\mathbb{C} = \{a + ib : a, b \in \mathbb{R}\}.$$

Now we introduce some basic notions.

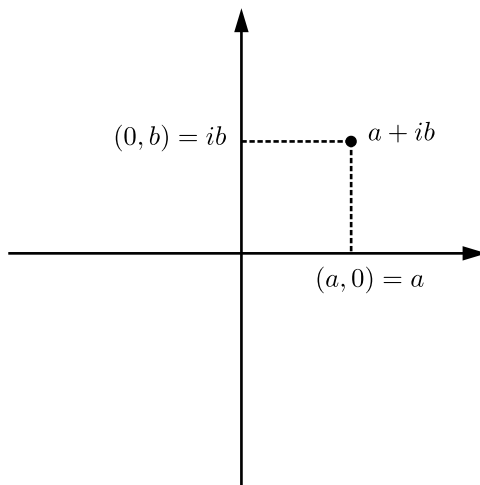


Figure 1.2 Real part and imaginary part

Definition 1.4

Given $z = a + ib \in \mathbb{C}$, the real number a is called the *real part* of z and the real number b is called the *imaginary part* of z (see Figure 1.2). We also write

$$a = \operatorname{Re} z \quad \text{and} \quad b = \operatorname{Im} z.$$

Example 1.5

If $z = 2 + i3$, then $\operatorname{Re} z = 2$ and $\operatorname{Im} z = 3$.

Two complex numbers $z_1, z_2 \in \mathbb{C}$ are equal if and only if

$$\operatorname{Re} z_1 = \operatorname{Re} z_2 \quad \text{and} \quad \operatorname{Im} z_1 = \operatorname{Im} z_2.$$

Definition 1.6

Given $z \in \mathbb{C}$ in the form

$$z = r \cos \theta + ir \sin \theta, \tag{1.3}$$

with $r \geq 0$ and $\theta \in \mathbb{R}$, the number r is called the *modulus* of z and the number θ is called an *argument* of z (see Figure 1.3). We also write

$$r = |z| \quad \text{and} \quad \theta = \arg z.$$

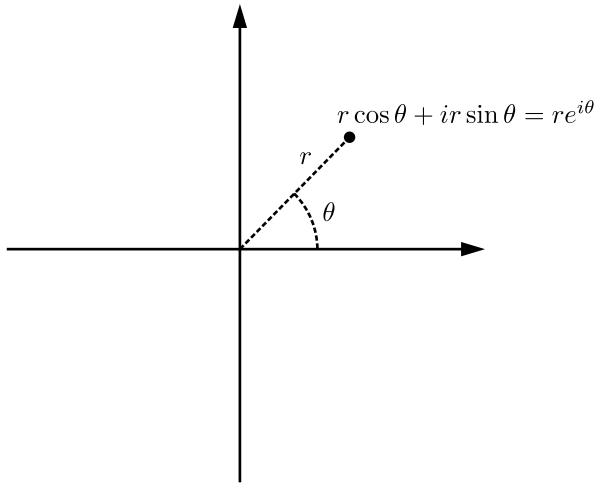


Figure 1.3 Modulus, argument and polar form

We emphasize that the number θ in (1.3) is not unique. Indeed, if identity (1.3) holds, then

$$z = r \cos(\theta + 2k\pi) + ir \sin(\theta + 2k\pi) \quad \text{for } k \in \mathbb{Z}.$$

One can easily establish the following result.

Proposition 1.7

If $z = a + ib \in \mathbb{C}$, then

$$|z| = \sqrt{a^2 + b^2} \quad (1.4)$$

and

$$\arg z = \begin{cases} \tan^{-1}(b/a) & \text{if } a > 0, \\ \pi/2 & \text{if } a = 0 \text{ and } b > 0, \\ \tan^{-1}(b/a) + \pi & \text{if } a < 0, \\ -\pi/2 & \text{if } a = 0 \text{ and } b < 0, \end{cases} \quad (1.5)$$

where \tan^{-1} is the inverse of the tangent with values in the interval $(-\pi/2, \pi/2)$.

It follows from (1.4) that

$$|\operatorname{Re} z| \leq |z| \quad \text{and} \quad |\operatorname{Im} z| \leq |z|. \quad (1.6)$$

Example 1.8

If $z = 2 + i2\sqrt{3}$, then

$$|z| = \sqrt{2^2 + 2^2 \cdot 3} = \sqrt{16} = 4,$$

and using the first branch in (1.5), we obtain

$$\arg z = \tan^{-1} \frac{2\sqrt{3}}{2} = \tan^{-1} \sqrt{3} = \frac{\pi}{3}.$$

The following result is a simple consequence of Definition 1.6.

Proposition 1.9

Two complex numbers $z, w \in \mathbb{C}$ are equal if and only if $|z| = |w|$ and

$$\arg z - \arg w = 2k\pi \quad \text{for some } k \in \mathbb{Z}.$$

1.2 Polar Form

It is often useful to write a complex number in the form (1.3) or also in the following alternative form.

Definition 1.10

Given $z \in \mathbb{C}$ in the form $z = r \cos \theta + ir \sin \theta$, with $r \geq 0$ and $\theta \in \mathbb{R}$, we write

$$z = re^{i\theta} = |z|e^{i \arg z}.$$

We say that $z = a + ib$ is the *Cartesian form* of z and that $z = re^{i\theta}$ is the *polar form* of z .

Example 1.11

If $z = 1 + i$, then

$$|z| = \sqrt{2} \quad \text{and} \quad \arg z = \tan^{-1} 1 = \pi/4.$$

Hence, the polar form of z is $\sqrt{2}e^{i\pi/4}$.

Now we describe the product and the quotient of complex numbers in terms of the polar form.

Proposition 1.12

If $z_1 = r_1e^{i\theta_1}$ and $z_2 = r_2e^{i\theta_2}$, then

$$z_1 z_2 = r_1 r_2 e^{i(\theta_1 + \theta_2)} \tag{1.7}$$

and

$$\frac{z_1}{z_2} = \frac{r_1}{r_2} e^{i(\theta_1 - \theta_2)} \quad \text{for } z_2 \neq 0.$$

Proof

For the product, by (1.3) we have

$$z_1 z_2 = (r_1 \cos \theta_1 + ir_1 \sin \theta_1)(r_2 \cos \theta_2 + ir_2 \sin \theta_2),$$

and thus,

$$\begin{aligned}
 z_1 z_2 &= r_1 r_2 (\cos \theta_1 + i \sin \theta_1)(\cos \theta_2 + i \sin \theta_2) \\
 &= r_1 r_2 (\cos \theta_1 \cos \theta_2 - \sin \theta_1 \sin \theta_2) \\
 &\quad + i r_1 r_2 (\cos \theta_1 \sin \theta_2 + \sin \theta_1 \cos \theta_2) \\
 &= r_1 r_2 \cos(\theta_1 + \theta_2) + i r_1 r_2 \sin(\theta_1 + \theta_2) \\
 &= r_1 r_2 e^{i(\theta_1 + \theta_2)}.
 \end{aligned} \tag{1.8}$$

For the quotient, we note that if $w = \rho e^{i\alpha}$ is a complex number satisfying $w z_2 = z_1$, then it follows from (1.8) that

$$w z_2 = \rho r_2 e^{i(\alpha + \theta_2)} = r_1 e^{i\theta_1}.$$

By Proposition 1.9, we obtain

$$\rho r_2 = r_1 \quad \text{and} \quad \alpha + \theta_2 - \theta_1 = 2k\pi$$

for some $k \in \mathbb{Z}$. Therefore,

$$\frac{z_1}{z_2} = w = \rho e^{i\alpha} = \frac{r_1}{r_2} e^{i(\theta_2 - \theta_1 + 2k\pi)} = \frac{r_1}{r_2} e^{i(\theta_2 - \theta_1)}$$

for $z_2 \neq 0$, which yields the desired statement. \square

Now we consider the powers and the roots of complex numbers, also expressed in terms of the polar form. For the powers, the following result is an immediate consequence of (1.7).

Proposition 1.13

If $z = r e^{i\theta}$ and $k \in \mathbb{N}$, then $z^k = r^k e^{ik\theta}$.

The roots of complex numbers require some extra care.

Proposition 1.14

If $z = r e^{i\theta}$ and $k \in \mathbb{N}$, then the complex numbers w such that $w^k = z$ are given by

$$w = r^{1/k} e^{i(\theta + 2\pi j)/k}, \quad j = 0, 1, \dots, k-1. \tag{1.9}$$

Proof

If $w = \rho e^{i\alpha}$ satisfies $w^k = z$, then it follows from Proposition 1.13 that

$$w^k = \rho^k e^{ik\alpha} = r e^{i\theta}.$$

By Proposition 1.9, we obtain $\rho^k = r$ and $k\alpha - \theta = 2\pi j$ for some $j \in \mathbb{Z}$. Therefore,

$$w = \rho e^{i\alpha} = r^{1/k} e^{i(\theta+2\pi j)/k},$$

and the distinct values of $e^{i(\theta+2\pi j)/k}$ are obtained for $j \in \{0, 1, \dots, k-1\}$. \square

We note that the roots in (1.9) of the complex number z are uniformly distributed on the circle of radius $r^{1/k}$ centered at the origin.

Example 1.15

For $k = 5$ the roots of 1 are

$$w = 1^{1/5} e^{i(0+2\pi j)/5} = e^{i2\pi j/5}, \quad j = 0, 1, 2, 3, 4$$

(see Figure 1.4).

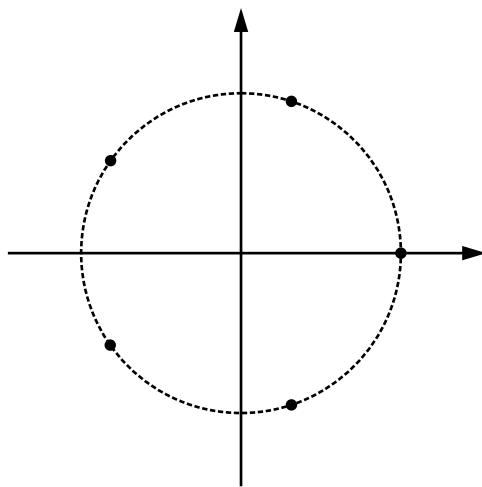


Figure 1.4 Roots of 1 for $k = 5$

1.3 Conjugate

Now we introduce the notion of the conjugate of a complex number.

Definition 1.16

Given $z = a + ib \in \mathbb{C}$, the complex number $\bar{z} = a - ib$ is called the *conjugate* of z (see Figure 1.5).

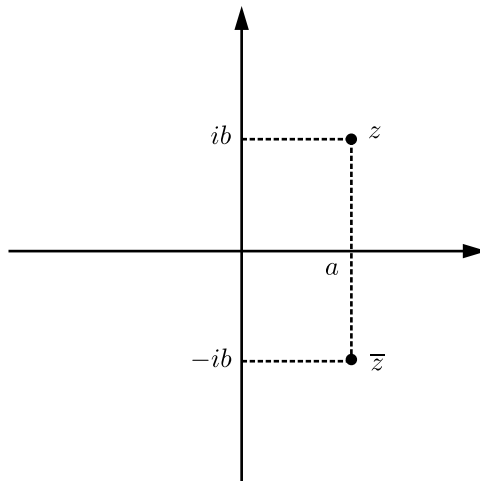


Figure 1.5 \bar{z} is the conjugate of z

Clearly, $\overline{\bar{z}} = z$. Moreover, if $z = re^{i\theta}$, then

$$\begin{aligned}\bar{z} &= \overline{r \cos \theta + ir \sin \theta} \\ &= r \cos \theta - ir \sin \theta \\ &= r \cos(-\theta) + ir \sin(-\theta) = re^{-i\theta}.\end{aligned}$$

Proposition 1.17

For every $z \in \mathbb{C}$, we have $z\bar{z} = |z|^2$.

Proof

Given a complex number $z = re^{i\theta}$, we have

$$z\bar{z} = re^{i\theta}re^{-i\theta} = r^2e^{i0} = |z|^2.$$

This yields the desired identity. \square

Proposition 1.18

For every $z, w \in \mathbb{C}$, we have

$$\overline{z+w} = \bar{z} + \bar{w} \quad \text{and} \quad \overline{zw} = \bar{z}\bar{w}.$$

Proof

Let $z = a + ib$ and $w = c + id$, with $a, b, c, d \in \mathbb{R}$. Taking conjugates, we obtain

$$\bar{z} = a - ib \quad \text{and} \quad \bar{w} = c - id.$$

Therefore,

$$\bar{z} + \bar{w} = (a + c) - i(b + d) \tag{1.10}$$

(see Figure 1.6). On the other hand,

$$z + w = (a + c) + i(b + d),$$

and thus,

$$\overline{z+w} = (a + c) - i(b + d). \tag{1.11}$$

The identity $\overline{z+w} = \bar{z} + \bar{w}$ now follows readily from (1.10) and (1.11).

Moreover, if $z = re^{i\theta}$ and $w = \rho e^{i\alpha}$, then

$$zw = r\rho e^{i(\theta+\alpha)},$$

and thus,

$$\overline{zw} = r\rho e^{-i(\theta+\alpha)} = re^{-i\theta}\rho e^{-i\alpha} = \bar{z}\bar{w}.$$

This completes the proof of the proposition. \square

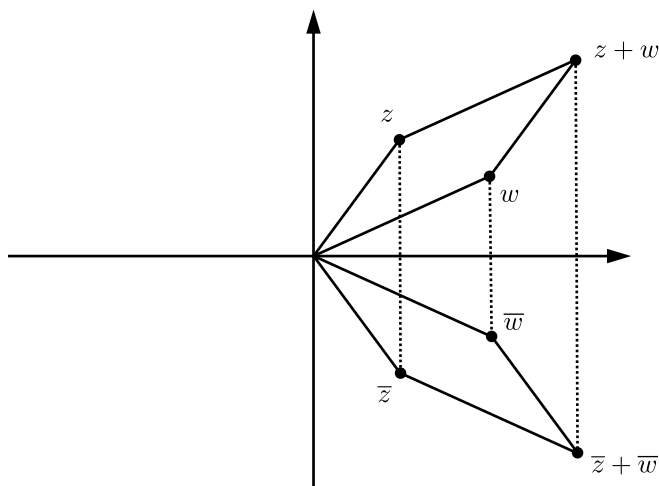


Figure 1.6 Points z , w , $z + w$ and their conjugates

Example 1.19

Let us consider the polynomial

$$p(z) = \sum_{k=0}^n a_k z^k$$

for some real numbers $a_k \in \mathbb{R}$. We have $\overline{a_k} = a_k$ for each k , and thus,

$$\begin{aligned} \overline{p(z)} &= \sum_{k=0}^n \overline{a_k z^k} = \sum_{k=0}^n \overline{a_k} \overline{z^k} \\ &= \sum_{k=0}^n a_k \overline{z^k} = p(\overline{z}). \end{aligned}$$

In particular, if $p(z) = 0$ for some $z \in \mathbb{C}$, then

$$p(\overline{z}) = \overline{p(z)} = \overline{0} = 0.$$

This implies that the nonreal roots of p occur in pairs of conjugates.

We also use the notion of conjugate to establish the following result.

Proposition 1.20

For every $z, w \in \mathbb{C}$, we have:

1. $|z| \geq 0$, and $|z| = 0$ if and only if $z = 0$;
2. $|zw| = |z| \cdot |w|$;
3. $|z + w| \leq |z| + |w|$.

Proof

The first property follows immediately from (1.4). For the second property, we note that

$$\begin{aligned} |zw|^2 &= zw\overline{z\overline{w}} = zw\overline{z}\overline{\overline{w}} \\ &= z\overline{z}w\overline{w} = |z|^2|w|^2 \end{aligned} \quad (1.12)$$

for every $z, w \in \mathbb{C}$. Finally, for the third property, we observe that

$$\begin{aligned} |z + w|^2 &= (z + w)(\overline{z + w}) \\ &= (z + w)(\overline{z} + \overline{w}) \\ &= z\overline{z} + z\overline{w} + w\overline{z} + w\overline{w} \\ &= |z|^2 + |w|^2 + 2\operatorname{Re}(z\overline{w}). \end{aligned}$$

It follows from (1.6) and (1.12) that

$$\operatorname{Re}(z\overline{w}) \leq |z\overline{w}| = |z| \cdot |\overline{w}| = |z| \cdot |w|,$$

and hence,

$$\begin{aligned} |z + w|^2 &\leq |z|^2 + |w|^2 + 2|z| \cdot |w| \\ &= (|z| + |w|)^2. \end{aligned}$$

This completes the proof of the proposition. \square

1.4 Complex Functions

In this section we consider complex-valued functions of a complex variable. Given a set $\Omega \subset \mathbb{C}$, a function $f: \Omega \rightarrow \mathbb{C}$ can be written in the form

$$f(x + iy) = u(x, y) + iv(x, y),$$

with $u(x, y), v(x, y) \in \mathbb{R}$ for each $x + iy \in \Omega$. In fact, since the set of complex numbers \mathbb{C} is identified with \mathbb{R}^2 , we obtain functions $u, v: \Omega \rightarrow \mathbb{R}$.

Definition 1.21

The function u is called the *real part* of f and the function v is called the *imaginary part* of f .

Example 1.22

For $f(z) = z^2$, we have

$$f(x + iy) = (x + iy)^2 = x^2 - y^2 + i2xy,$$

and hence,

$$u(x, y) = x^2 - y^2 \quad \text{and} \quad v(x, y) = 2xy.$$

Example 1.23

For $f(z) = z^3$, we have

$$f(x + iy) = (x + iy)^3 = x^3 - 3xy^2 + i(3x^2y - y^3),$$

and hence,

$$u(x, y) = x^3 - 3xy^2 \quad \text{and} \quad v(x, y) = 3x^2y - y^3.$$

Now we introduce various complex functions.

Definition 1.24

We define the *exponential* of the complex number $z = x + iy$ by

$$e^z = e^x(\cos y + i \sin y).$$

Example 1.25

For each $z = x + i0 \in \mathbb{R}$, we have

$$e^z = e^x(\cos 0 + i \sin 0) = e^x(1 + i0) = e^x.$$

Hence, the exponential of a real number x coincides with the exponential of x when this is seen as a complex number.

Example 1.26

For $z = i\pi$, we have

$$e^{i\pi} = e^{0+i\pi} = e^0(\cos \pi + i \sin \pi) = 1(-1 + i0) = -1.$$

We also describe several properties of the exponential.

Proposition 1.27

For every $z, w \in \mathbb{C}$ and $k \in \mathbb{Z}$, we have:

1. $e^{z+w} = e^z e^w$ and $1/e^z = e^{-z}$;
2. $\overline{e^z} = e^{\bar{z}}$;
3. $(e^z)^k = e^{kz}$;
4. $e^{z+i2k\pi} = e^z$.

Proof

Given $z = x + iy$ and $w = x' + iy'$, we have

$$\begin{aligned} e^{z+w} &= e^{(x+x')+(y+y')} \\ &= e^{x+x'} [\cos(y+y') + i \sin(y+y')] \\ &= e^x e^{x'} [(\cos y \cos y' - \sin y \sin y') + i(\sin y \cos y' + \sin y' \cos y)] \\ &= e^x e^{x'} (\cos y + i \sin y)(\cos y' + i \sin y') \\ &= e^x (\cos y + i \sin y) e^{x'} (\cos y' + i \sin y') \\ &= e^z e^w. \end{aligned}$$

In particular,

$$e^z e^{-z} = e^{z-z} = e^0 = 1,$$

and thus $1/e^z = e^{-z}$. This establishes the first property. For the second, we note that

$$\begin{aligned} \overline{e^z} &= \overline{e^x \cos y + i e^x \sin y} \\ &= e^x \cos y - i e^x \sin y = e^x (\cos y - i \sin y) \\ &= e^x [\cos(-y) + i \sin(-y)] = e^{x-iy} = e^{\bar{z}}. \end{aligned}$$

The third property follows from the first one by induction, and for the fourth we note that

$$\begin{aligned} e^{z+i2\pi k} &= e^{x+i(y+2k\pi)} \\ &= e^x [\cos(y+2k\pi) + i \sin(y+2k\pi)] \\ &= e^x (\cos y + i \sin y) = e^z. \end{aligned}$$

This completes the proof of the proposition. \square

Now we consider the trigonometric functions.

Definition 1.28

The *cosine* and the *sine* of $z \in \mathbb{C}$ are defined respectively by

$$\cos z = \frac{e^{iz} + e^{-iz}}{2}$$

and

$$\sin z = \frac{e^{iz} - e^{-iz}}{2i}.$$

Example 1.29

For $z = x + i0 \in \mathbb{R}$, we have

$$\begin{aligned} \cos z &= \frac{e^{ix} + e^{-ix}}{2} \\ &= \frac{1}{2}(\cos x + i \sin x + \cos x - i \sin x) = \cos x \end{aligned}$$

and

$$\begin{aligned} \sin z &= \frac{e^{ix} - e^{-ix}}{2i} \\ &= \frac{1}{2i}(\cos x + i \sin x - \cos x + i \sin x) = \sin x. \end{aligned}$$

Hence, the cosine and the sine of a real number x coincide respectively with the cosine and the sine of x when this is seen as a complex number.

Example 1.30

For $z = iy$, we have

$$\cos(iy) = \frac{e^{-y} + e^y}{2}.$$

In particular, the cosine is not a bounded function in \mathbb{C} , in contrast to what happens in \mathbb{R} . One can show in a similar manner that the sine is also unbounded in \mathbb{C} .

Example 1.31

Let us solve the equation $\cos z = 1$, that is,

$$\frac{e^{iz} + e^{-iz}}{2} = 1.$$

For $w = e^{iz}$, we have $1/w = e^{-iz}$, and thus,

$$w + \frac{1}{w} = 2,$$

that is, $w^2 - 2w + 1 = 0$. This yields $w = 1$, which is the same as $e^{iz} = 1$. Writing $z = x + iy$, with $x, y \in \mathbb{R}$, we obtain

$$\begin{aligned} e^{iz} &= e^{i(x+iy)} = e^{-y+ix} \\ &= e^{-y} \cos x + ie^{-y} \sin x, \end{aligned}$$

and it follows from $e^{iz} = 1 + i0$ that

$$e^{-y} \cos x = 1 \quad \text{and} \quad e^{-y} \sin x = 0.$$

Since $e^{-y} \neq 0$, we obtain $\sin x = 0$. Together with the identity $\cos^2 x + \sin^2 x = 1$, this yields $\cos x = \pm 1$. But since $e^{-y} > 0$, it follows from $e^{-y} \cos x = 1$ that $\cos x = 1$, and hence, $e^{-y} = 1$. Therefore, $x = 2k\pi$, with $k \in \mathbb{Z}$, and $y = 0$. The solution of the equation $\cos z = 1$ is thus $z = 2k\pi$, with $k \in \mathbb{Z}$.

The following result is an immediate consequence of Proposition 1.27.

Proposition 1.32

For every $z \in \mathbb{C}$ and $k \in \mathbb{Z}$, we have

$$\cos(z + 2k\pi) = \cos z \quad \text{and} \quad \sin(z + 2k\pi) = \sin z.$$

- [read online Companions of Paradise online](#)
- [download online Lion of Liberty: Patrick Henry and the Call to a New Nation](#)
- [read OpenLayers 3: Beginner's Guide](#)
- [download online Daddy's Little Helper \(Midnight Sons, Book 3\) pdf, azw \(kindle\), epub, doc, mobi](#)
- **read On Populist Reason**
- [The Past: A Novel book](#)

- <http://crackingscience.org/?library/The-Psychology-of-Wealth--Understand-Your-Relationship-with-Money-and-Achieve-Prosperity.pdf>
- <http://honareavalmusic.com/?books/The-New-Soft-War-on-Women--How-the-Myth-of-Female-Ascendance-Is-Hurting-Women--Men-and-Our-Economy.pdf>
- <http://www.mmastyles.com/books/OpenLayers-3--Beginner-s-Guide.pdf>
- <http://www.satilik-kopek.com/library/Discrete-Mathematics--Universitext-.pdf>
- <http://twilightblogs.com/library/Military-Strategy-and-Operational-Art--Joining-the-Fray--Outside-Military-Intervention-in-Civil-Wars.pdf>
- <http://hasanetmekci.com/ebooks/The-Past--A-Novel.pdf>