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Complex Analysis and Differential Equations

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Complex Analysis and Differential Equations

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Preface

This book is essentially two books in one. Namely, it is an introduction to two large areas of mathematics—*complex analysis* and *differential equations*—and the material is naturally divided into two parts. This includes holomorphic functions, analytic functions, ordinary differential equations, Fourier series, and partial differential equations. Moreover, half of the book consists of approximately 200 worked-out problems plus 200 exercises of variable level of difficulty. The worked-out problems fill the gap between the theory and the exercises.

To a considerable extent, the parts of complex analysis and differential equations can be read independently. In the second part, some special emphasis is given to the applications of complex analysis to differential equations. On the other hand, the material is still developed with sufficient detail in order that the book contains an ample introduction to differential equations, and not strictly related to complex analysis.

The text is tailored to any course giving a first introduction to complex analysis or to differential equations, assuming as prerequisite only a basic knowledge of linear algebra and of differential and integral calculus. But it can also be used for independent study. In particular, the book contains a large number of examples illustrating the new concepts and results. Moreover, the worked-out problems, carefully prepared for each part of the theory, make this the ideal book for independent study, allowing the student to actually see how the theory applies, before solving the exercises.

Lisbon, Portugal — Luis Barreira and Claudia Valls

Contents

Part I
Complex Analysis

1
Basic Notions

In this chapter we introduce the set of complex numbers, as well as some basic notions. In particular, we describe the operations of addition and multiplication, as well as the powers and roots of complex numbers. We also introduce various complex functions that are natural extensions of corresponding functions in the real case, such as the exponential, the cosine, the sine, and the logarithm.

1.1 Complex Numbers

We first introduce the set of complex numbers as the set of pairs of real numbers equipped with operations of addition and multiplication.

Definition 1.1

The set $\mathbb{C}$ of *complex numbers* is the set $\mathbb{R}^2$ of pairs of real numbers equipped with the operations

$$(a,b)+(c,d)=(a+c,b+d) \tag{1.1}$$

and

$$(a,b)\cdot(c,d)=(ac-bd,ad+bc) \tag{1.2}$$

for each $(a,b),(c,d)\in\mathbb{R}^2$.

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One can easily verify that the operations of addition and multiplication in (1.1) and (1.2) are commutative, that is,

$$(a,b)+(c,d)=(c,d)+(a,b)$$

and

$$(a,b)\cdot(c,d)=(c,d)\cdot(a,b)$$

for every $(a,b),(c,d)\in\mathbb{R}^2$.

Example 1.2

For example, we have

$$(5,4)+(3,2)=(8,6)$$

and

$$(2,1)\cdot(-1,6)=\big(2\cdot(-1)-1\cdot 6, 2\cdot 6+1\cdot(-1)\big)=(-8,11).$$

For simplicity of notation, we always write

$$(a,0)=a,$$

thus identifying the pair $(a,0)\in\mathbb{R}^2$ with the real number a (see Figure 1.1). We define the *imaginary unit* by

$$(0,1)=i$$

(see Figure 1.1).

Proposition 1.3

We have $i^2=-1$ and $a+ib=(a,b)$ for every $a,b\in\mathbb{R}$.

Proof

Indeed,

$$i^2=(0,1)\cdot(0,1)=(-1,0)=-1,$$

and

$$\begin{aligned} a+ib&=(a,0)+(0,1)\cdot(b,0)\\ &=(a,0)+(0,b)=(a,b), \end{aligned}$$

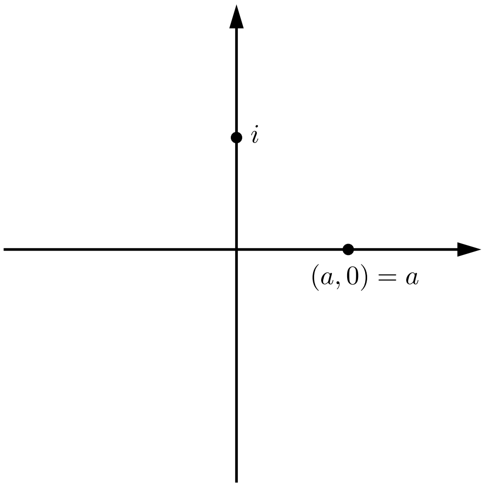

Figure 1.1 Real number a and imaginary unit i

which yields the desired statement. □

We thus have

$$\mathbb{C} = \{a + ib : a, b \in \mathbb{R}\}.$$

Now we introduce some basic notions.

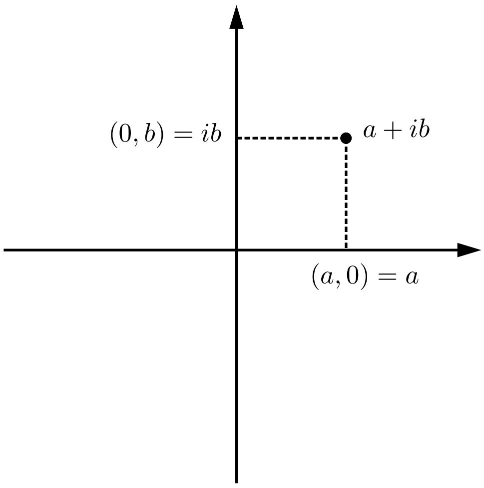

Figure 1.2 Real part and imaginary part

Definition 1.4

Given $z = a + ib \in \mathbb{C}$, the real number a is called the *real part* of z and the real number b is called the *imaginary part* of z (see Figure 1.2). We also write

$$a = \operatorname{Re} z \quad \text{and} \quad b = \operatorname{Im} z.$$

Example 1.5

If $z = 2 + i3$, then $\operatorname{Re} z = 2$ and $\operatorname{Im} z = 3$.

Two complex numbers $z_1, z_2 \in \mathbb{C}$ are equal if and only if

$$\operatorname{Re} z_1 = \operatorname{Re} z_2 \quad \text{and} \quad \operatorname{Im} z_1 = \operatorname{Im} z_2.$$

Definition 1.6

Given $z \in \mathbb{C}$ in the form

$$z = r\cos\theta + ir\sin\theta, \tag{1.3}$$

with $r \geq 0$ and $\theta \in \mathbb{R}$, the number r is called the *modulus* of z and the number θ is called an *argument* of z (see Figure 1.3). We also write

$$r = |z| \quad \text{and} \quad \theta = \arg z.$$

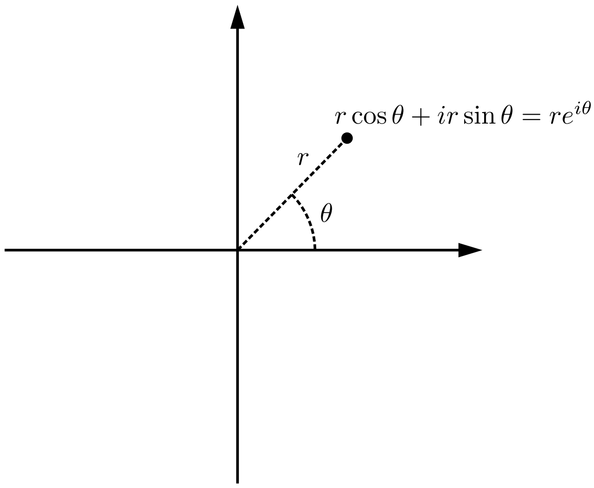

Figure 1.3 Modulus, argument and polar form

We emphasize that the number θ in (1.3) is not unique. Indeed, if identity (1.3) holds, then

$$z = r\cos(\theta + 2k\pi) + ir\sin(\theta + 2k\pi) \quad \text{for } k \in \mathbb{Z}.$$

One can easily establish the following result.

Proposition 1.7

If $z = a + ib \in \mathbb{C}$, then

$$|z| = \sqrt{a^2 + b^2} \tag{1.4}$$

and

$$\arg z = \begin{cases} \tan^{-1}(b/a) & \text{if } a > 0, \\ \pi/2 & \text{if } a = 0 \text{ and } b > 0, \\ \tan^{-1}(b/a) + \pi & \text{if } a < 0, \\ -\pi/2 & \text{if } a = 0 \text{ and } b < 0, \end{cases} \tag{1.5}$$

where $\tan^{-1}$ is the inverse of the tangent with values in the interval $(-\pi/2, \pi/2)$.

It follows from (1.4) that

$$|\operatorname{Re} z| \le |z| \quad \text{and} \quad |\operatorname{Im} z| \le |z|. \tag{1.6}$$

Example 1.8

If $z = 2 + i2\sqrt{3}$, then

$$|z| = \sqrt{2^2 + 2^2 \cdot 3} = \sqrt{16} = 4,$$

and using the first branch in (1.5), we obtain

$$\arg z = \tan^{-1} \frac{2\sqrt{3}}{2} = \tan^{-1} \sqrt{3} = \frac{\pi}{3}.$$

The following result is a simple consequence of Definition 1.6.

Proposition 1.9

Two complex numbers $z, w \in \mathbb{C}$ are equal if and only if $|z| = |w|$ and

$$\arg z - \arg w = 2k\pi \quad \text{for some } k \in \mathbb{Z}.$$

1.2 Polar Form

It is often useful to write a complex number in the form (1.3) or also in the following alternative form.

Definition 1.10

Given $z \in \mathbb{C}$ in the form $z = r\cos\theta + ir\sin\theta$, with $r \geq 0$ and $\theta \in \mathbb{R}$, we write

$$z = re^{i\theta} = |z|e^{i\arg z}.$$

We say that $z = a + ib$ is the *Cartesian form* of z and that $z = re^{i\theta}$ is the *polar form* of z.

Example 1.11

If $z = 1 + i$, then

$$|z| = \sqrt{2} \quad \text{and} \quad \arg z = \tan^{-1} 1 = \pi/4.$$

Hence, the polar form of z is $\sqrt{2}e^{i\pi/4}$.

Now we describe the product and the quotient of complex numbers in terms of the polar form.

Proposition 1.12

If $z_1 = r_1e^{i\theta_1}$ and $z_2 = r_2e^{i\theta_2}$, then

$$z_1z_2 = r_1r_2e^{i(\theta_1+\theta_2)} \tag{1.7}$$

and

$$\frac{z_1}{z_2} = \frac{r_1}{r_2}e^{i(\theta_1-\theta_2)} \quad \text{for} \quad z_2 \neq 0.$$

Proof

For the product, by (1.3) we have

$$z_1z_2 = (r_1\cos\theta_1 + ir_1\sin\theta_1)(r_2\cos\theta_2 + ir_2\sin\theta_2),$$

and thus,

$$\begin{aligned} z_1 z_2 &= r_1 r_2 (\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2) \\ &= r_1 r_2 (\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2) \\ &\quad + i r_1 r_2 (\cos\theta_1\sin\theta_2 + \sin\theta_1\cos\theta_2) \\ &= r_1 r_2 \cos(\theta_1+\theta_2) + i r_1 r_2 \sin(\theta_1+\theta_2) \\ &= r_1 r_2 e^{i(\theta_1+\theta_2)}. \end{aligned} \tag{1.8}$$

For the quotient, we note that if $w = \rho e^{i\alpha}$ is a complex number satisfying $wz_2 = z_1$, then it follows from (1.8) that

$$wz_2 = \rho r_2 e^{i(\alpha+\theta_2)} = r_1 e^{i\theta_1}.$$

By Proposition 1.9, we obtain

$$\rho r_2 = r_1 \quad \text{and} \quad \alpha + \theta_2 - \theta_1 = 2k\pi$$

for some $k \in \mathbb{Z}$. Therefore,

$$\frac{z_1}{z_2} = w = \rho e^{i\alpha} = \frac{r_1}{r_2} e^{i(\theta_2-\theta_1+2k\pi)} = \frac{r_1}{r_2} e^{i(\theta_2-\theta_1)}$$

for $z_2 \neq 0$, which yields the desired statement. □

Now we consider the powers and the roots of complex numbers, also expressed in terms of the polar form. For the powers, the following result is an immediate consequence of (1.7).

Proposition 1.13

If $z = re^{i\theta}$ and $k \in \mathbb{N}$, then $z^k = r^k e^{ik\theta}$.

The roots of complex numbers require some extra care.

Proposition 1.14

If $z = re^{i\theta}$ and $k \in \mathbb{N}$, then the complex numbers w such that $w^k = z$ are given by

$$w = r^{1/k} e^{i(\theta+2\pi j)/k}, \quad j = 0, 1, \ldots, k-1. \tag{1.9}$$

Proof

If $w = \rho e^{i\alpha}$ satisfies $w^k = z$, then it follows from Proposition 1.13 that

$$w^k = \rho^k e^{ik\alpha} = re^{i\theta}.$$

By Proposition 1.9, we obtain $\rho^k = r$ and $k\alpha - \theta = 2\pi j$ for some $j \in \mathbb{Z}$. Therefore,

$$w = \rho e^{i\alpha} = r^{1/k} e^{i(\theta+2\pi j)/k},$$

and the distinct values of $e^{i(\theta+2\pi j)/k}$ are obtained for $j \in \{0, 1, \dots, k-1\}$. $\square$

We note that the roots in (1.9) of the complex number z are uniformly distributed on the circle of radius $r^{1/k}$ centered at the origin.

Example 1.15

For $k = 5$ the roots of 1 are

$$w = 1^{1/5} e^{i(0+2\pi j)/5} = e^{i2\pi j/5}, \quad j = 0, 1, 2, 3, 4$$

(see Figure 1.4).

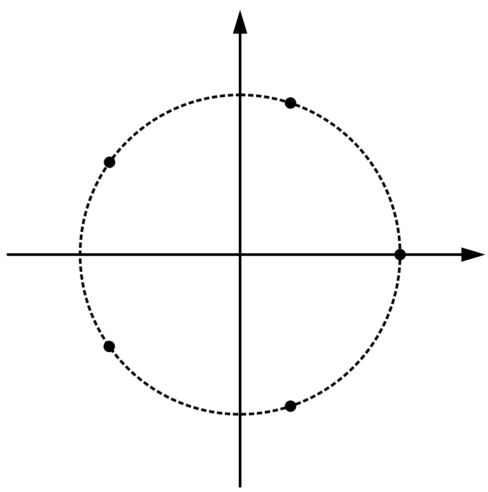

Figure 1.4 Roots of 1 for $k = 5$

1.3 Conjugate

Now we introduce the notion of the conjugate of a complex number.

Definition 1.16

Given $z = a + ib \in \mathbb{C}$, the complex number $\overline{z} = a - ib$ is called the *conjugate* of z (see Figure 1.5).

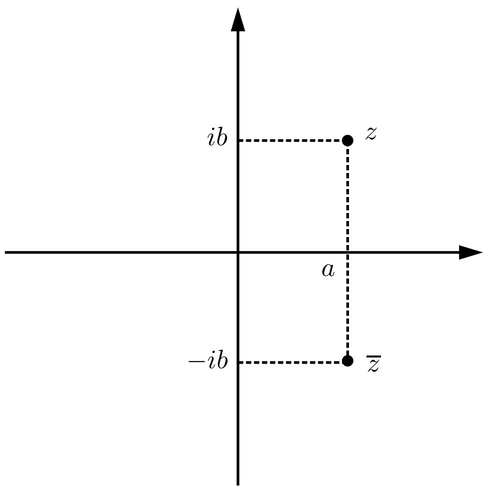

Figure 1.5 $\overline{z}$ is the conjugate of z

Clearly, $\overline{\overline{z}} = z$. Moreover, if $z = re^{i\theta}$, then

$$\begin{aligned}\overline{z} &= \overline{r\cos\theta + ir\sin\theta} \\ &= r\cos\theta - ir\sin\theta \\ &= r\cos(-\theta) + ir\sin(-\theta) = re^{-i\theta}.\end{aligned}$$

Proposition 1.17

For every $z \in \mathbb{C}$, we have $z\overline{z} = |z|^2$.

Proof

Given a complex number $z = re^{i\theta}$, we have

$$z\overline{z} = re^{i\theta} re^{-i\theta} = r^2 e^{i0} = |z|^2.$$

This yields the desired identity. □

Proposition 1.18

For every $z, w \in \mathbb{C}$, we have

$$\overline{z+w} = \overline{z} + \overline{w} \quad \text{and} \quad \overline{zw} = \overline{z}\,\overline{w}.$$

Proof

Let $z = a + ib$ and $w = c + id$, with $a, b, c, d \in \mathbb{R}$. Taking conjugates, we obtain

$$\overline{z} = a - ib \quad \text{and} \quad \overline{w} = c - id.$$

Therefore,

$$\overline{z} + \overline{w} = (a + c) - i(b + d) \tag{1.10}$$

(see Figure 1.6). On the other hand,

$$z + w = (a + c) + i(b + d),$$

and thus,

$$\overline{z+w} = (a + c) - i(b + d). \tag{1.11}$$

The identity $\overline{z+w} = \overline{z} + \overline{w}$ now follows readily from (1.10) and (1.11).

Moreover, if $z = re^{i\theta}$ and $w = \rho e^{i\alpha}$, then

$$zw = r\rho e^{i(\theta+\alpha)},$$

and thus,

$$\overline{zw} = r\rho e^{-i(\theta+\alpha)} = re^{-i\theta}\rho e^{-i\alpha} = \overline{z}\,\overline{w}.$$

This completes the proof of the proposition. □

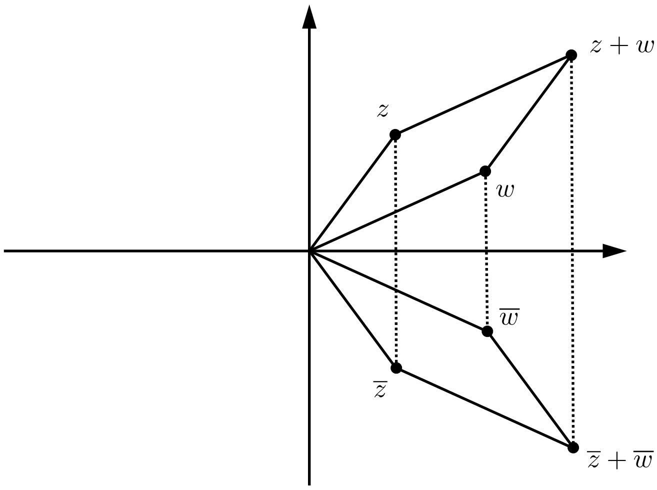

Figure 1.6 Points z, w, $z+w$ and their conjugates

Example 1.19

Let us consider the polynomial

$$p(z) = \sum_{k=0}^{n} a_k z^k$$

for some real numbers $a_k \in \mathbb{R}$. We have $\overline{a_k} = a_k$ for each k, and thus,

$$\begin{aligned}\overline{p(z)} &= \sum_{k=0}^{n} \overline{a_k z^k} = \sum_{k=0}^{n} \overline{a_k}\,\overline{z^k} \\ &= \sum_{k=0}^{n} a_k \overline{z}^k = p(\overline{z}).\end{aligned}$$

In particular, if $p(z) = 0$ for some $z \in \mathbb{C}$, then

$$p(\overline{z}) = \overline{p(z)} = \overline{0} = 0.$$

This implies that the nonreal roots of p occur in pairs of conjugates.

We also use the notion of conjugate to establish the following result.

Proposition 1.20

For every $z, w \in \mathbb{C}$, we have:

1. $|z| \geq 0$, and $|z| = 0$ if and only if $z = 0$;
2. $|zw| = |z| \cdot |w|$;
3. $|z + w| \leq |z| + |w|$.

Proof

The first property follows immediately from (1.4). For the second property, we note that

$$\begin{aligned} |zw|^2 &= zw\overline{zw} = zw\overline{z}\,\overline{w} \\ &= z\overline{z}w\overline{w} = |z|^2|w|^2 \end{aligned} \tag{1.12}$$

for every $z, w \in \mathbb{C}$. Finally, for the third property, we observe that

$$\begin{aligned} |z + w|^2 &= (z + w)(\overline{z + w}) \\ &= (z + w)(\overline{z} + \overline{w}) \\ &= z\overline{z} + z\overline{w} + w\overline{z} + w\overline{w} \\ &= |z|^2 + |w|^2 + 2\,\mathrm{Re}(z\overline{w}). \end{aligned}$$

It follows from (1.6) and (1.12) that

$$\mathrm{Re}(z\overline{w}) \leq |z\overline{w}| = |z| \cdot |\overline{w}| = |z| \cdot |w|,$$

and hence,

$$\begin{aligned} |z + w|^2 &\leq |z|^2 + |w|^2 + 2|z| \cdot |w| \\ &= \big(|z| + |w|\big)^2. \end{aligned}$$

This completes the proof of the proposition. $\square$

1.4 Complex Functions

In this section we consider complex-valued functions of a complex variable. Given a set $\Omega \subset \mathbb{C}$, a function $f \colon \Omega \to \mathbb{C}$ can be written in the form

$$f(x + iy) = u(x, y) + iv(x, y),$$

with $u(x,y), v(x,y) \in \mathbb{R}$ for each $x+iy \in \Omega$. In fact, since the set of complex numbers $\mathbb{C}$ is identified with $\mathbb{R}^2$, we obtain functions $u, v \colon \Omega \to \mathbb{R}$.

Definition 1.21

The function u is called the *real part* of f and the function v is called the *imaginary part* of f.

Example 1.22

For $f(z) = z^2$, we have

$$f(x+iy) = (x+iy)^2 = x^2 - y^2 + i2xy,$$

and hence,

$$u(x,y) = x^2 - y^2 \quad \text{and} \quad v(x,y) = 2xy.$$

Example 1.23

For $f(z) = z^3$, we have

$$f(x+iy) = (x+iy)^3 = x^3 - 3xy^2 + i\big(3x^2y - y^3\big),$$

and hence,

$$u(x,y) = x^3 - 3xy^2 \quad \text{and} \quad v(x,y) = 3x^2y - y^3.$$

Now we introduce various complex functions.

Definition 1.24

We define the *exponential* of the complex number $z = x+iy$ by

$$e^z = e^x(\cos y + i \sin y).$$

Example 1.25

For each $z = x + i0 \in \mathbb{R}$, we have

$$e^z = e^x(\cos 0 + i \sin 0) = e^x(1 + i0) = e^x.$$

Hence, the exponential of a real number x coincides with the exponential of x when this is seen as a complex number.

Example 1.26

For $z = i\pi$, we have

$$e^{i\pi} = e^{0+i\pi} = e^0(\cos\pi + i\sin\pi) = 1(-1+i0) = -1.$$

We also describe several properties of the exponential.

Proposition 1.27

For every $z, w \in \mathbb{C}$ and $k \in \mathbb{Z}$, we have:
1. $e^{z+w} = e^z e^w$ and $1/e^z = e^{-z}$;
2. $\overline{e^z} = e^{\overline{z}}$;
3. $(e^z)^k = e^{kz}$;
4. $e^{z+i2k\pi} = e^z$.

Proof

Given $z = x + iy$ and $w = x' + iy'$, we have

$$\begin{aligned}
e^{z+w} &= e^{(x+x')+i(y+y')} \\
&= e^{x+x'}\big[\cos(y+y') + i\sin(y+y')\big] \\
&= e^x e^{x'}\big[(\cos y\cos y' - \sin y\sin y') + i(\sin y\cos y' + \sin y'\cos y)\big] \\
&= e^x e^{x'}(\cos y + i\sin y)(\cos y' + i\sin y') \\
&= e^x(\cos y + i\sin y)e^{x'}(\cos y' + i\sin y') \\
&= e^z e^w.
\end{aligned}$$

In particular,

$$e^z e^{-z} = e^{z-z} = e^0 = 1,$$

and thus $1/e^z = e^{-z}$. This establishes the first property. For the second, we note that

$$\begin{aligned}
\overline{e^z} &= \overline{e^x\cos y + ie^x\sin y} \\
&= e^x\cos y - ie^x\sin y = e^x(\cos y - i\sin y) \\
&= e^x\big[\cos(-y) + i\sin(-y)\big] = e^{x-iy} = e^{\overline{z}}.
\end{aligned}$$

The third property follows from the first one by induction, and for the fourth we note that

$$\begin{aligned}e^{z+i2\pi k} &= e^{x+i(y+2k\pi)}\\ &= e^x\bigl[\cos(y+2k\pi)+i\sin(y+2k\pi)\bigr]\\ &= e^x(\cos y+i\sin y)=e^z.\end{aligned}$$

This completes the proof of the proposition. □

Now we consider the trigonometric functions.

Definition 1.28

The *cosine* and the *sine* of $z \in \mathbb{C}$ are defined respectively by

$$\cos z = \frac{e^{iz}+e^{-iz}}{2}$$

and

$$\sin z = \frac{e^{iz}-e^{-iz}}{2i}.$$

Example 1.29

For $z = x + i0 \in \mathbb{R}$, we have

$$\begin{aligned}\cos z &= \frac{e^{ix}+e^{-ix}}{2}\\ &= \frac{1}{2}(\cos x + i\sin x + \cos x - i\sin x) = \cos x\end{aligned}$$

and

$$\begin{aligned}\sin z &= \frac{e^{ix}-e^{-ix}}{2i}\\ &= \frac{1}{2i}(\cos x + i\sin x - \cos x + i\sin x) = \sin x.\end{aligned}$$

Hence, the cosine and the sine of a real number x coincide respectively with the cosine and the sine of x when this is seen as a complex number.

Example 1.30

For $z = iy$, we have

$$\cos(iy) = \frac{e^{-y} + e^{y}}{2}.$$

In particular, the cosine is not a bounded function in $\mathbb{C}$, in contrast to what happens in $\mathbb{R}$. One can show in a similar manner that the sine is also unbounded in $\mathbb{C}$.

Example 1.31

Let us solve the equation $\cos z = 1$, that is,

$$\frac{e^{iz} + e^{-iz}}{2} = 1.$$

For $w = e^{iz}$, we have $1/w = e^{-iz}$, and thus,

$$w + \frac{1}{w} = 2,$$

that is, $w^2 - 2w + 1 = 0$. This yields $w = 1$, which is the same as $e^{iz} = 1$. Writing $z = x + iy$, with $x, y \in \mathbb{R}$, we obtain

$$\begin{aligned} e^{iz} &= e^{i(x+iy)} = e^{-y+ix} \\ &= e^{-y}\cos x + ie^{-y}\sin x, \end{aligned}$$

and it follows from $e^{iz} = 1 + i0$ that

$$e^{-y}\cos x = 1 \quad \text{and} \quad e^{-y}\sin x = 0.$$

Since $e^{-y} \neq 0$, we obtain $\sin x = 0$. Together with the identity $\cos^2 x + \sin^2 x = 1$, this yields $\cos x = \pm 1$. But since $e^{-y} > 0$, it follows from $e^{-y}\cos x = 1$ that $\cos x = 1$, and hence, $e^{-y} = 1$. Therefore, $x = 2k\pi$, with $k \in \mathbb{Z}$, and $y = 0$. The solution of the equation $\cos z = 1$ is thus $z = 2k\pi$, with $k \in \mathbb{Z}$.

The following result is an immediate consequence of Proposition 1.27.

Proposition 1.32

For every $z \in \mathbb{C}$ and $k \in \mathbb{Z}$, we have

$$\cos(z + 2k\pi) = \cos z \quad \text{and} \quad \sin(z + 2k\pi) = \sin z.$$