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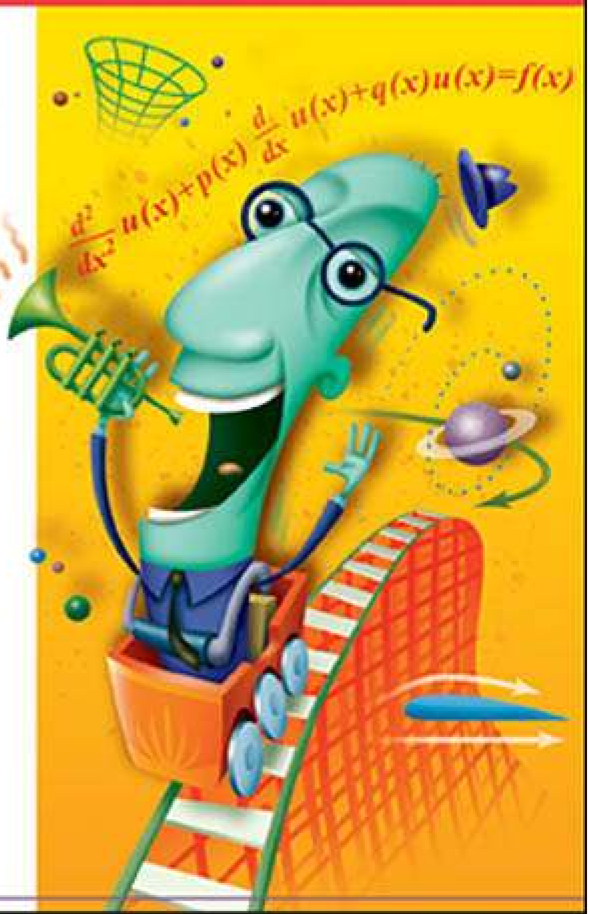

Steven G. Krantz

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PREFACE

If calculus is the heart of modern science, then differential equations are its guts. All physical laws, from the motion of a vibrating string to the orbits of the planets to Einstein's field equations, are expressed in terms of differential equations. Classically, ordinary differential equations described one-dimensional phenomena and partial differential equations described higher-dimensional phenomena. But, with the modern advent of dynamical systems theory, ordinary differential equations are now playing a role in the scientific analysis of phenomena in all dimensions.

Virtually every sophomore science student will take a course in introductory ordinary differential equations. Such a course is often fleshed out with a brief look at the Laplace transform, Fourier series, and boundary value problems for the Laplacian. Thus the student gets to see a little advanced material, and some higher-dimensional ideas, as well.

As indicated in the first paragraph, differential equations is a lovely venue for mathematical modeling and the applications of mathematical thinking. Truly meaningful and profound ideas from physics, engineering, aeronautics, statics, mechanics, and other parts of physical science are beautifully illustrated with differential equations.

We propose to write a text on ordinary differential equations that will be meaningful, accessible, and engaging for a student with a basic grounding in calculus (for example, the student who has studied *Calculus Demystified* by this author will be more than ready for *Differential Equations Demystified*). There will be many applications, many graphics, a plethora of worked examples, and hundreds of stimulating exercises. The student who completes this book will be ready to go on to advanced analytical work in applied mathematics, engineering, and other fields of mathematical science. It will be a powerful and useful learning tool.

Steven G. Krantz

CHAPTER 1

What Is a Differential Equation?

1.1 Introductory Remarks

A *differential equation* is an equation relating some function f to one or more of its derivatives. An example is

$$\frac{d^2 f}{dx^2} + 2x\frac{df}{dx} + f^2(x) = \sin x. \tag{1}$$

Observe that this particular equation involves a function f together with its first and second derivatives. The objective in solving an equation like (1) is to *find the* *function* f. Thus we already perceive a fundamental new paradigm: When we solve an algebraic equation, we seek a number or perhaps a collection of numbers; but when we solve a differential equation we seek one or more *functions*.

Many of the laws of nature—in physics, in engineering, in chemistry, in biology, and in astronomy—find their most natural expression in the language of differential equations. Put in other words, differential equations are the language of nature. Applications of differential equations also abound in mathematics itself, especially in geometry and harmonic analysis and modeling. Differential equations occur in economics and systems science and other fields of mathematical science.

It is not difficult to perceive why differential equations arise so readily in the sciences. If $y = f(x)$ is a given function, then the derivative df/dx can be interpreted as the rate of change of f with respect to x. In any process of nature, the variables involved are related to their rates of change by the basic scientific principles that govern the process—that is, by the laws of nature. When this relationship is expressed in mathematical notation, the result is usually a differential equation.

Certainly Newton's Law of Universal Gravitation, Maxwell's field equations, the motions of the planets, and the refraction of light are important physical examples which can be expressed using differential equations. Much of our understanding of nature comes from our ability to solve differential equations. The purpose of this book is to introduce you to some of these techniques.

The following example will illustrate some of these ideas. According to Newton's second law of motion, the acceleration a of a body of mass m is proportional to the total force F acting on the body. The standard implementation of this relationship is

$$F = m \cdot a. \tag{2}$$

Suppose in particular that we are analyzing a falling body of mass m. Express the height of the body from the surface of the Earth as $y(t)$ feet at time t. The only force acting on the body is that due to gravity. If g is the acceleration due to gravity (about -32 ft/sec^2 near the surface of the Earth) then the force exerted on the body is $m \cdot g$. And of course the acceleration is d^2y/dt^2. Thus Newton's law (2) becomes

$$m \cdot g = m \cdot \frac{d^2y}{dt^2} \tag{3}$$

or

$$g = \frac{d^2y}{dt^2}.$$

We may make the problem a little more interesting by supposing that air exerts a resisting force proportional to the velocity. If the constant of proportionality is k, then the total force acting on the body is $mg - k \cdot (dy/dt)$. Then the equation (3) becomes

$$m \cdot g - k \cdot \frac{dy}{dt} = m \cdot \frac{d^2y}{dt^2}. \tag{4}$$

Equations (3) and (4) express the essential attributes of this physical system.

A few additional examples of differential equations are these:

$$(1-x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + p(p+1)y = 0; \tag{5}$$

$$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + (x^2 - p^2)y = 0; \tag{6}$$

$$\frac{d^2y}{dx^2} + xy = 0; \tag{7}$$

$$(1-x^2)y'' - xy' + p^2y = 0; \tag{8}$$

$$y'' - 2xy' + 2py = 0; \tag{9}$$

$$\frac{dy}{dx} = k \cdot y. \tag{10}$$

Equations (5)–(9) are called Legendre's equation, Bessel's equation, Airy's equation, Chebyshev's equation, and Hermite's equation respectively. Each has a vast literature and a history reaching back hundreds of years. We shall touch on each of these equations later in the book. Equation (10) is the equation of exponential decay (or of biological growth).

Math Note: A great many of the laws of nature are expressed as second-order differential equations. This fact is closely linked to Newton's second law, which expresses force as mass time acceleration (and acceleration is a *second derivative*). But some physical laws are given by higher-order equations. The Euler–Bernoulli beam equation is fourth-order.

Each of equations (5)–(9) is of second-order, meaning that the highest derivative that appears is the second. Equation (10) is of first-order, meaning that the highest derivative that appears is the first. Each equation is an *ordinary differential equation*, meaning that it involves a function of a single variable and the *ordinary derivatives* (*not* partial derivatives) of that function.

1.2 The Nature of Solutions

An ordinary differential equation of order n is an equation involving an unknown function f together with its derivatives

$$\frac{df}{dx}, \frac{d^2 f}{dx^2}, \ldots, \frac{d^n f}{dx^n}.$$

We might, in a more formal manner, express such an equation as

$$F\left(x, y, \frac{df}{dx}, \frac{d^2 f}{dx^2}, \ldots, \frac{d^n f}{dx^n}\right) = 0.$$

How do we verify that a given function f is actually the solution of such an equation?

The answer to this question is best understood in the context of concrete examples.

e.g.

EXAMPLE 1.1

Consider the differential equation

$$y'' - 5y' + 6y = 0.$$

Without saying how the solutions are actually *found*, we can at least check that $y_1(x) = e^{2x}$ and $y_2(x) = e^{3x}$ are both solutions.

To verify this assertion, we note that

$$\begin{aligned} y_1'' - 5y_1' + 6y_1 &= 2 \cdot 2 \cdot e^{2x} - 5 \cdot 2 \cdot e^{2x} + 6 \cdot e^{2x} \\ &= [4 - 10 + 6] \cdot e^{2x} \\ &\equiv 0 \end{aligned}$$

and

$$\begin{aligned} y_2'' - 5y_2' + 6y_2 &= 3 \cdot 3 \cdot e^{3x} - 5 \cdot 3 \cdot e^{3x} + 6 \cdot e^{3x} \\ &= [9 - 15 + 6] \cdot e^{3x} \\ &\equiv 0. \end{aligned}$$

This process, of verifying that a *function* is a solution of the given differential equation, is most likely entirely new for you. You will want to practice and become accustomed to it. In the last example, you may check that any function of the form

$$y(x) = c_1 e^{2x} + c_2 e^{3x} \tag{1}$$

(where c_1, c_2 are arbitrary constants) is also a solution of the differential equation.

Math Note: This last observation is an instance of the principle of superposition in physics. Mathematicians refer to the algebraic operation in equation (1) as "taking a linear combination of solutions" while physicists think of the process as superimposing forces.

An important obverse consideration is this: When you are going through the procedure to solve a differential equation, how do you know when you are finished? The answer is that the solution process is complete when all derivatives have been eliminated from the equation. For then you will have y expressed in terms of x (at least implicitly). Thus you will have found the sought-after function.

For a large class of equations that we shall study in detail in the present book, we will find a number of "independent" solutions equal to the order of the differential equation. Then we will be able to form a so-called "general solution" by combining them as in (1). Of course we shall provide all the details of this process in the development below.

You Try It: Verify that each of the functions $y_1(x) = e^x$ $y_2(x) = e^{2x}$ and $y_3(x) = e^{-4x}$ is a solution of the differential equation

$$\frac{d^3y}{dx^3} + \frac{d^2y}{dx^2} - 10\frac{dy}{dx} + 8y = 0.$$

More generally, check that $y(x) = c_1e^x + c_2e^{2x} + c_3e^{-4x}$ (where c_1, c_2, c_3 are arbitrary constants) is a "general solution" of the differential equation.

Sometimes the solution of a differential equation will be expressed as an *implicitly defined function*. An example is the equation

$$\frac{dy}{dx} = \frac{y^2}{1 - xy}, \tag{2}$$

which has solution

$$xy = \ln y + c. \tag{3}$$

Equation (3) represents a solution because all derivatives have been eliminated.

Example 1.2 below contains the details of the verification that (3) is the solution of (2).

Math Note: It takes some practice to get used to the idea that an implicitly defined function is still a function. A classic and familiar example is the equation

$$x^2 + y^2 = 1. \tag{4}$$

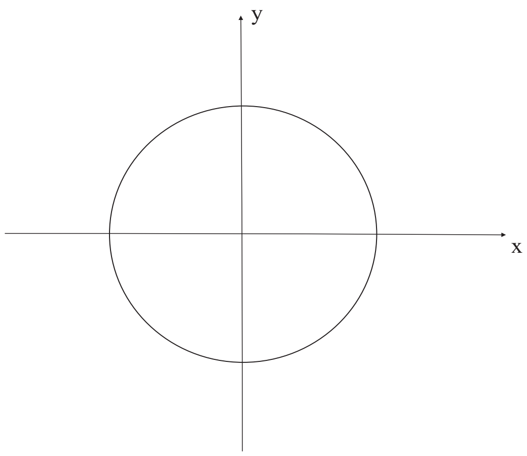

Fig. 1.1.

This relation expresses y as a function of x at most points. Refer to Fig. 1.1. In fact the equation (4) entails

$$y = +\sqrt{1 - x^2}$$

when y is positive and

$$y = -\sqrt{1 - x^2}$$

when y is negative. It is only at the exceptional points $(-1, 0)$ and $(-1, 0)$, where the tangent lines are vertical, that y *cannot* be expressed as a function of x.

Note here that the hallmark of what we call a *solution* is that it has no derivatives in it: it is a straightforward formula, relating y (the dependent variable) to x (the independent variable).

e.g.

EXAMPLE 1.2

To verify that (3) is indeed a solution of (2), let us differentiate:

$$\frac{d}{dx}[xy] = \frac{d}{dx}[\ln y + c],$$

hence

$$1 \cdot y + x \cdot \frac{dy}{dx} = \frac{dy/dx}{y}$$

or

$$\frac{dy}{dx}\left[\frac{1}{y} - x\right] = y.$$

In conclusion,

$$\frac{dy}{dx} = \frac{y^2}{1 - xy},$$

as desired.

One unifying feature of the two examples that we have now seen of verifying solutions is this: When we solve an equation of order n, we expect n "independent solutions" (we shall have to say later just what this word "independent" means) and we expect n undetermined constants. In the first example, the equation was of order 2 and the undetermined constants were c_1 and c_2. In the second example, the equation was of order 1 and the undetermined constant was c.

You Try It: Verify that the equation $x \sin y = \cos y$ gives an implicit solution to the differential equation

$$\frac{dy}{dx}[x \cot y + 1] = -1.$$

1.3 Separable Equations

In this section we shall encounter our first general class of equations with the property that

(i) We can immediately recognize members of this class of equations.
(ii) We have a simple and direct method for (in principle)[1] solving such equations.

This is the class of *separable equations*.

DEFINITION 1.1
An ordinary differential equation is *separable* if it is possible, by elementary algebraic manipulation, to arrange the equation so that all the dependent variables (usually the y variable) are on one side and all the independent variables

[1]We throw in this caveat because it can happen, and frequently does happen, that we can write down integrals that represent solutions of our differential equation, but *we are unable to evaluate those integrals*. This is annoying, but we shall later—in Chapter 7—learn numerical techniques that will address such an impasse.

(usually the x variable) are on the other side. The corresponding solution technique is called *separation of variables*.

Let us learn the method by way of some examples.

e.g. **EXAMPLE 1.3**
Solve the ordinary differential equation

$$y' = 2xy.$$

SOLUTION
In the method of separation of variables—which is a method for *first-order* equations only—it is useful to write the derivative using Leibniz notation. Thus we have

$$\frac{dy}{dx} = 2xy.$$

We rearrange this equation as

$$\frac{dy}{y} = 2x\,dx.$$

[It should be noted here that we use the shorthand dy to stand for $\frac{dy}{dx}\,dx$.]

Now we can integrate both sides of the last displayed equation to obtain

$$\int \frac{dy}{y} = \int 2x\,dx.$$

We are fortunate in that both integrals are easily evaluated. We obtain

$$\ln y = x^2 + c.$$

[It is important here that we include the constant of integration. We combine the constant from the left-hand integral and the constant from the right-hand integral into a single constant c.] Thus

$$y = e^{x^2+c}.$$

We may abbreviate e^c by D and rewrite this last equation as

$$y = De^{x^2}. \qquad (1)$$

Notice two important features of our final representation for the solution:

(i) We have re-expressed the constant e^c as the positive constant D.
(ii) Our solution contains one free constant, as we may have anticipated since the differential equation is of order 1.

We invite you to verify that the solution in equation (1) actually satisfies the original differential equation.

EXAMPLE 1.4 *e.g.*
Solve the differential equation

$$xy' = (1 - 2x^2) \tan y.$$

SOLUTION
We first write the equation in Leibniz notation. Thus

$$x \cdot \frac{dy}{dx} = (1 - 2x^2) \tan y.$$

Separating variables, we find that

$$\cot y \, dy = \left[\frac{1}{x} - 2x \right] dx.$$

Applying the integral to both sides gives

$$\int \cot y \, dy = \int \left[\frac{1}{x} - 2x \right] dx$$

or

$$\ln \sin y = \ln x - x^2 + C.$$

Again note that we were careful to include a constant of integration.
We may express our solution as

$$\sin y = e^{\ln x - x^2 + C}$$

or

$$\sin y = D \cdot x \cdot e^{-x^2}.$$

The result may be written as

$$y = \sin^{-1} \left[D \cdot x \cdot e^{-x^2} \right].$$

We invite you to verify that this is indeed a solution to the given differential equation.

Math Note: It should be stressed that not all ordinary differential equations are separable. As an instance, the equation

$$x^2y + y^2x = \sin(xy)$$

cannot be separated so that all the x's are on one side of the equation and all the y's on the other side.

You Try It: Use the method of separation of variables to solve the differential equation

$$x^3y' = y.$$

1.4 First-Order Linear Equations

Another class of differential equations that is easily recognized and readily solved (at least in principle) is that of first-order linear equations.

DEFINITION 1.2
An equation is said to be *first-order linear* if it has the form

$$y' + a(x)y = b(x). \tag{1}$$

The "first-order" aspect is obvious: only first derivatives appear in the equation. The "linear" aspect depends on the fact that the left-hand side involves a differential operator that acts linearly on the space of differentiable functions. Roughly speaking, a differential equation is linear if y and its derivatives are not multiplied together, not raised to powers, and do not occur as the arguments of functions. This is an advanced idea that we shall explicate in detail later. For now, you should simply accept that an equation of the form (1) is first-order linear, and that we will soon have a recipe for solving it.

As usual, we explicate the method by proceeding directly to the examples.

e.g.

EXAMPLE 1.5
Consider the differential equation

$$y' + 2xy = x.$$

Find a complete solution.

SOLUTION
This equation is plainly not separable (try it and convince yourself that this is so). Instead we endeavor to multiply both sides of the equation by some function that will make each side readily integrable. It turns out that there is a trick that always works: You multiply both sides by $e^{\int a(x)\,dx}$.

Like many tricks, this one may seem unmotivated. But let us try it out and see how it works in practice. Now

$$\int a(x)\,dx = \int 2x\,dx = x^2.$$

[At this point we *could* include a constant of integration, but it is not necessary.] Thus $e^{\int a(x)\,dx} = e^{x^2}$. Multiplying both sides of our equation by this factor gives

$$e^{x^2} \cdot y' + e^{x^2} \cdot 2xy = e^{x^2} \cdot x$$

or

$$\left[e^{x^2} \cdot y\right]' = x \cdot e^{x^2}.$$

It is the last step that is a bit tricky. For a first-order linear equation, it is *guaranteed* that if we multiply through by $e^{\int a(x)\,dx}$ then the left-hand side of the equation will end up being the derivative of $[e^{\int a(x)\,dx} \cdot y]$. Now of course we integrate both sides of the equation:

$$\int \left[e^{x^2} \cdot y\right]' dx = \int x \cdot e^{x^2}\,dx.$$

We can perform both the integrations: on the left-hand side we simply apply the fundamental theorem of calculus; on the right-hand side we do the integration. The result is

$$e^{x^2} \cdot y = \frac{1}{2} \cdot e^{x^2} + C$$

or

$$y = \frac{1}{2} + Ce^{-x^2}.$$

Observe that, as we usually expect, the solution has one free constant (because the original differential equation was of order 1). We invite you to check that this solution actually satisfies the differential equation.

Math Note: Of course not all ordinary differential equations are first order linear. The equation

$$[y']^2 - y = \sin x$$

is indeed first order—because the highest derivative that appears is the first derivative. But it is *non*linear because the function y' is multiplied by itself. The equation

$$y'' \cdot y - y' = e^x$$

is second order and is also nonlinear—because y'' is multiplied times y.

Summary of the method of first-order linear equations

To solve a first-order linear equation

$$y' + a(x)y = b(x),$$

multiply both sides of the equation by the "integrating factor" $e^{\int a(x)\,dx}$ and then integrate.

EXAMPLE 1.6
Solve the differential equation

$$x^2y' + xy = x^2 \cdot \sin x.$$

SOLUTION
First observe that this equation is not in the standard form (equation (1)) for first-order linear. We render it so by multiplying through by a factor of $1/x^2$. Thus the equation becomes

$$y' + \frac{1}{x}y = \sin x.$$

Now $a(x) = 1/x$, $\int a(x)\,dx = \ln|x|$, and $e^{\int a(x)\,dx} = |x|$. We multiply the differential equation through by this factor. In fact, in order to simplify the calculus, we shall restrict attention to $x > 0$. Thus we may eliminate the absolute value signs.

Thus

$$xy' + y = x \cdot \sin x.$$