Brief Edition

Susanna S. Epp

Discrete Mathematics

An Introduction to Mathematical Reasoning

List of Symbols

Subject	Symbol	Meaning	Page
Logic	$\sim p$	not p	25
	$p \wedge q$	p and q	25
	$p \vee q$	p or q	25
	$p \oplus q$ or p XOR q	p or q but not both p and q	28
	$P \equiv Q$	P is logically equivalent to Q	30
	$p \to q$	if p then q	39
	$p \leftrightarrow q$	p if and only if q	44
	$\therefore$	therefore	50
	$P(x)$	predicate in x	64
	$P(x) \Rightarrow Q(x)$	every element in the truth set for $P(x)$ is in the truth set for $Q(x)$	71
	$P(x) \Leftrightarrow Q(x)$	$P(x)$ and $Q(x)$ have identical truth sets	71
	$\forall$	for all	68
	$\exists$	there exists	70
Number Theory	$d \mid n$	d divides n	134
	$d \nmid n$	d does not divide n	136
	n *div* d	the integer quotient of n divided by d	145
	n *mod* d	the integer remainder of n divided by d	145
	$\lvert x \rvert$	the absolute value of x	151
	$\gcd(a, b)$	the greatest common divisor of a and b	387
	$x \cong y$	x is approximately equal to y	181
Sequences	$\ldots$	and so forth	171
	$\sum_{k=m}^{n} a_k$	the summation from k equals m to n of a_k	174
	$\prod_{k=m}^{n} a_k$	the product from k equals m to n of a_k	177
	$n!$	n factorial	181
Set Theory	$a \in A$	a is an element of A	7
	$a \notin A$	a is not an element of A	7
	$\{a_1, a_2, \ldots, a_n\}$	the set with elements $a_1, a_2, \ldots, a_n$	7
	$\{x \in D \mid P(x)\}$	the set of all x in D for which $P(x)$ is true	8
	$\mathbf{R}, \mathbf{R}^-, \mathbf{R}^+, \mathbf{R}^{nonneg}$	the sets of all real numbers, negative real numbers, positive real numbers, and nonnegative real numbers	7, 8
	$\mathbf{Z}, \mathbf{Z}^-, \mathbf{Z}^+, \mathbf{Z}^{nonneg}$	the sets of all integers, negative integers, positive integers, and nonnegative integers	7, 8
	$\mathbf{Q}, \mathbf{Q}^-, \mathbf{Q}^+, \mathbf{Q}^{nonneg}$	the sets of all rational numbers, negative rational numbers, positive rational numbers, and nonnegative rational numbers	7, 8
	$\mathbf{N}$	the set of natural numbers	8
	$A \subseteq B$	A is a subset of B	9
	$A \nsubseteq B$	A is not a subset of B	9
	$A = B$	A equals B	252
	$A \cup B$	A union B	254
	$A \cap B$	A intersect B	254
	$B - A$	the difference of B minus A	254
	A^c	the complement of A	254
	(x, y)	ordered pair	11
	$(x_1, x_2, \ldots, x_n)$	ordered n-tuple	259
	$A \times B$	the Cartesian product of A and B	12
	$A_1 \times A_2 \times \cdots \times A_n$	the Cartesian product of $A_1, A_2, \ldots, A_n$	260
	$\emptyset$	the empty set	274
	$\mathscr{P}(A)$	the power set of A	259

Subject	Symbol	Meaning	Page
Counting and Probability	$N(A)$	the number of elements in set A	405
	$P(A)$	the probability of a set A	405
	$P(n, r)$	the number of r-permutations of a set of n elements	418
	$\binom{n}{r}$	n choose r, the number of r-combinations of a set of n elements, the number of r-element subsets of a set of n elements	182, 447
	ϵ	the null string	414
Functions	$f\colon X \to Y$	f is a function from X to Y	294
	$f(x)$	the value of f at x	294
	$x \xrightarrow{f} y$	f sends x to y	294
	$f(A)$	the image of A	305
	$f^{-1}(C)$	the inverse image of C	305
	I_x	the identity function on X	297
	b^x	b raised to the power x	312
	$\exp_b(x)$	b raised to the power x	312
	$\log_b(x)$	logarithm with base b of x	299
	F^{-1}	the inverse function of F	317
	$f \circ g$	the composition of g and f	322
Relations	$x\ R\ y$	x is related to y by R	14
	R^{-1}	the inverse relation of R	347
	$m \equiv n \pmod d$	m is congruent to n modulo d	363
	$[a]$	the equivalence class of a	364
	Z_n	the set of equivalence classes of integers modulo n	381
Graphs and Trees	$V(G)$	the set of vertices of a graph G	477
	$E(G)$	the set of edges of a graph G	477
	$\{v, w\}$	the edge joining v and w in a simple graph	483, 484
	K_n	complete graph on n vertices	484
	$K_{m,n}$	complete bipartite graph on (m, n) vertices	484
	$deg(v)$	degree of vertex v	486
	$v_0 e_1 v_1 e_2 \cdots e_n v_n$	a walk from v_0 to v_n	495

DISCRETE MATHEMATICS

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AN INTRODUCTION TO MATHEMATICAL REASONING

SUSANNA S. EPP
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Cover Photo: *The stones are discrete objects placed one on top of another like a chain of careful reasoning. A person who decides to build such a tower aspires to the heights and enjoys playing with a challenging problem. Choosing the stones takes both a scientific and an aesthetic sense. Getting them to balance requires patient effort and careful thought. And the tower that results is beautiful. A perfect metaphor for discrete mathematics!*

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Susanna S. Epp

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To my students, in appreciation for all they have taught me

CONTENTS

PREFACE

My purpose in developing this book was to provide a clear, accessible treatment of the essential aspects of discrete mathematics with a special focus on introducing students to mathematical proof. The book is based on the fourth edition of my Discrete Mathematics with applications, which has been used successfully by students at hundreds of institutions in North and South America, Europe, the Middle East, Asia, and Australia.

This version was written in response to requests by its users interested in a shorter, more streamlined treatment of the subject. Like its predecessor, however, its goal is to lay a mathematical foundation for upper-level courses in mathematics and computer science. The book may be used by students either before or after a course in calculus; a good background in algebra is the only prerequisite.

Recent curricular recommendations from the Institute for Electrical and Electronic Engineers Computer Society (IEEE-CS) and the Association for Computing Machinery (ACM) include discrete mathematics as the largest portion of "core knowledge" for computer science students and state that students should take at least a one-semester course in the subject as part of their first-year studies, with a two-semester course preferred when possible. This book includes most of the topics recommended by those organizations; the ones that it omits are often covered elsewhere in computer science programs. Coverage of sets, relations, and functions is foundational for all mathematics and computer science courses; inclusion of basic number theory provides background for students' future study of abstract algebra, certain computer algorithms, and cryptography; extensive work with quantifiers is especially useful for students who will take a course in real analysis or go on to study artificial intelligence, database theory, or techniques for establishing program correctness; and discussion of counting principles, graph theory, and the calculation of the likelihood of events prepares the way for further study of combinatorics and probability.

At one time, most of the topics in discrete mathematics were taught only to upper-level undergraduates. Discovering how to present these topics in ways that can be understood by first- and second-year students was the major and most interesting challenge of the work I have done. The presentation has been developed over a long period of experimentation during which my students were in many ways my teachers. Their questions, comments, and written work continue to show me what concepts and techniques cause them difficulty, and their reaction to my exposition shows me what works to build their understanding and to encourage their interest.

Themes of a Discrete Mathematics Course

Discrete mathematics describes processes that consist of a sequence of individual steps. This contrasts with calculus, which describes processes that change in a continuous fashion. Whereas the ideas of calculus were fundamental to the science and technology of the industrial revolution, the ideas of discrete mathematics underlie the science and technology of the computer age. Important themes of a first course in discrete mathematics are logic and proof, induction and recursion, discrete structures, and combinatorics and discrete probability.

Logic and Proof Probably the most important goal of a first course in discrete mathematics is to help students develop the ability to think abstractly. This means learning to use logically valid forms of argument and avoid common logical errors, appreciating what it means to reason from definitions, knowing how to use both direct and indirect argument to derive new results from those already known to be true, and being able to work with symbolic representations as if they were concrete objects.

Induction and Recursion An exciting development of recent years has been the increased appreciation for the power and beauty of "recursive thinking." To think recursively means to address a problem by assuming that similar problems of a smaller nature have already been solved and figuring out how to put those solutions together to solve the larger problem. Such thinking is used in modeling biological and financial systems, developing data management algorithms, and analyzing algorithms. Recurrence relations that result from recursive thinking often give rise to formulas that are verified by mathematical induction.

Discrete Structures Discrete mathematical structures are the abstract structures that describe, categorize, and reveal the underlying relationships among discrete mathematical objects. Those studied in this book are the sets of integers and rational numbers, general sets, functions, relations, and graphs and trees. In addition, the book includes brief introductions to Boolean algebras and commutative rings and fields.

Combinatorics and Discrete Probability Combinatorics is the mathematics of counting and arranging objects, and probability is the study of laws concerning the measurement of random or chance events. Discrete probability focuses on situations involving discrete sets of objects, such as finding the likelihood of obtaining a certain number of heads when an unbiased coin is tossed a certain number of times. Skill in using combinatorics and probability is needed in almost every discipline where mathematics is applied, from economics to biology, to computer science, to chemistry and physics, to business management.

Special Features of This Book

Mathematical Reasoning The feature that most distinguishes this book from other discrete mathematics texts is that it teaches—explicitly but in a way that is accessible to first- and second-year college and university students—the unspoken logic and reasoning that underlie mathematical thought. For many years I taught an intensively interactive transition-to-abstract-mathematics course to mathematics and computer science majors. This experience showed me that while it is possible to teach the majority of students to understand and construct straightforward mathematical arguments, the obstacles to doing so cannot be passed over lightly. To be successful, a text for such a course must address students' difficulties with logic and language directly and at some length. It must also include enough concrete examples and exercises to enable students to develop the mental models needed to conceptualize more abstract problems. The treatment of logic and proof in this book blends common sense and rigor in a way that explains the essentials, yet avoids overloading students with formal detail.

Spiral Approach to Concept Development A number of concepts in this book appear in increasingly more sophisticated forms in successive chapters to help students develop the ability to deal effectively with increasing levels of abstraction. For example, by the time students encounter the theory behind modular arithmetic and the solutions for linear Diophantine equations in Sections 8.4 and 8.5, they have been introduced to the logic of mathematical discourse in Chapters 1, 2, and 3, learned the basic methods of proof and the concepts of *mod* and *div* in Chapter 4, explored *mod* and *div* as functions in Chapter 7, and become familiar with equivalence relations in Sections 8.2 and 8.3. This approach builds in useful review and develops mathematical maturity in natural stages.

Support for the Student Students at colleges and universities inevitably have to learn a great deal on their own. Though it is often frustrating, learning to learn through self-study is a crucial step toward eventual success in a professional career. This book has a number of features to facilitate students' transition to independent learning.

Worked Examples
The book contains over 300 worked examples, which are written using a problem-solution format and are keyed in type and in difficulty to the exercises. Many solutions for the proof problems are developed in two stages: first a discussion of how one might come to think of the proof or disproof and then a summary of the solution, which is enclosed in a box. This format allows students to read the problem and skip immediately to the summary, if they wish, only going back to the discussion if they have trouble understanding the summary. The format also saves time for students who are rereading the text in preparation for an examination.

Marginal Notes and Test Yourself Questions
Notes about issues of particular importance and cautionary comments to help students avoid common mistakes are included in the margins throughout the book. Questions designed to focus attention on the main ideas of each section are located between the text and the exercises. For convenience, the questions use a fill-in-the-blank format, and the answers are found immediately after the exercises.

Exercises
The book contains almost 2000 exercises. The sets at the end of each section have been designed so that students with widely varying backgrounds and ability levels will find some exercises they can be sure to do successfully and also some exercises that will challenge them.

Solutions for Exercises
To provide adequate feedback for students between class sessions, Appendix B contains a large number of complete solutions to exercises. Students are strongly urged not to consult solutions until they have tried their best to answer the questions on their own. Once they have done so, however, comparing their answers with those given can lead to significantly improved understanding. In addition, many problems, including some of the most challenging, have partial solutions or hints so that students can determine whether they are on the right track and make adjustments if necessary. There are also plenty of exercises without solutions to help students learn to grapple with mathematical problems in a realistic environment.

Reference Features
My rationale for screening statements of definitions and theorems, for putting titles on exercises, and for giving the meanings of symbols and a list of reference formulas in the endpapers is to make it easier for students to use this book for review in a current course and as a reference in later ones. Figures and tables are included where doing so would help readers to a better understanding. In most, a second color is used to highlight meaning.

Support for the Instructor I have received a great deal of valuable feedback from instructors who have used editions of *Discrete Mathematics with Applications*, on which this book is based. Many aspects of this book have been improved through their suggestions. In addition to the following items, there is additional instructor support on the book's website, described later in the preface.

Exercises
The large variety of exercises at all levels of difficulty allows instructors great freedom to tailor a course to the abilities of their students. Exercises with solutions in the back of the book have numbers in blue, and those whose solutions are given in a separate *Student Solutions Manual and Study Guide* have numbers that are a multiple of three. There are exercises of every type represented in this book that have no answer in either location to enable instructors to assign whatever mixture they prefer of exercises with and without answers. The ample number of exercises of all kinds gives instructors a significant choice of problems to use for review assignments and exams. Instructors are invited to use the many exercises stated as questions rather than in "prove that" form to stimulate class discussion on the role of proof and counterexample in problem solving.

Flexible Sections
Most sections are divided into subsections so that an instructor who is pressed for time can choose to cover certain subsections only and either omit the rest or leave them for the students to study on their own. The division into subsections also makes it easier for instructors to break up sections if they wish to spend more then one day on them.

Presentation of Proof Methods
It is inevitable that the proofs and disproofs in this book will seem easy to instructors. Many students, however, find them difficult. In showing students how to discover and construct proofs and disproofs, I have tried to describe the kinds of approaches that mathematicians use when confronting challenging problems in their own research.

Instructor Solutions
Complete instructor solutions to all exercises are available to anyone teaching a course from this book via Cengage's Solution Builder service. Instructors can sign up for access at www.cengage.com/solutionbuilder.

Companion Website
www.cengage.com/math/epp

A website has been developed for this book that contains information and materials for both students and instructors. It includes:

- descriptions and links to many sites on the Internet with accessible information about discrete mathematical topics,
- links to applets that illustrate or provide practice in the concepts of discrete mathematics,
- additional examples and exercises with solutions,
- review guides for the chapters of the book.

A special section for instructors contains:

- suggestions about how to approach the material of each chapter,
- solutions for all exercises not fully solved in Appendix B,
- ideas for projects and writing assignments,
- PowerPoint slides,
- review sheets and additional exercises for quizzes and exams.

Student Solutions Manual and Study Guide

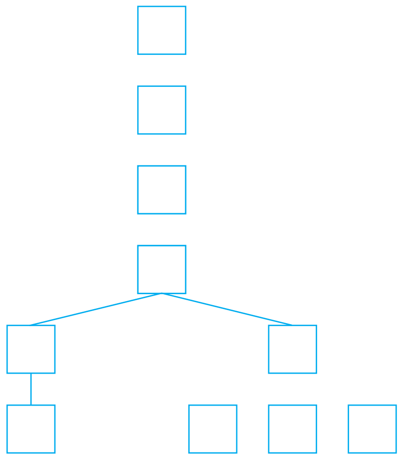
(ISBN-10: 0-495-82618-9; ISBN-13: 978-0-495-82618-7)

In writing this book, I strove to give sufficient help to students through the exposition in the text, the worked examples, and the exercise solutions, so that the book itself would provide all that a student would need to successfully master the material of the course. I believe that students who finish the study of this book with the ability to solve, on their own, all the exercises with full solutions in Appendix B will have developed an excellent command of the subject. Nonetheless, I have become aware that some students want the opportunity to obtain additional helpful materials. In response, I developed a Student Solutions Manual and Study Guide, available separately from this book, which contains complete solutions to every exercise that is not completely answered in Appendix B and whose number is divisible by 3. The guide also includes alternative explanations for some of the concepts and review questions for each chapter.

Organization

The following tree diagram shows, approximately, how the chapters of this book depend on each other. Chapters on different branches of the tree are sufficiently independent that instructors need to make at most minor adjustments if they skip chapters but follow paths along branches of the tree.

In most cases, covering only the core material in each chapter is adequate preparation for moving down the tree.



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By sharing their thoughts and thought processes with me, they taught me how to teach them better. I am very grateful for their help. I owe the DePaul University administration, especially my dean, Charles Suchar, and my former deans, Michael Mezey and Richard Meister, a special word of thanks for considering the writing of this book a worthwhile scholarly endeavor.

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The older I get the more I realize the profound debt I owe my own mathematics teachers for shaping the way I perceive the subject. My first thanks must go to my husband, Helmut Epp, who, on a high school date (!), introduced me to the power and beauty of the field axioms and the view that mathematics is a subject with ideas as well as formulas and techniques. In my formal education, I am most grateful to Daniel Zelinsky and Ky Fan at Northwestern University and Izaak Wirszup, I. N. Herstein, and Irving Kaplansky at the University of Chicago, all of whom, in their own ways, helped lead me to appreciate the elegance, rigor, and excitement of mathematics.

To my family, I owe thanks beyond measure. I am grateful to my mother, whose keen interest in the workings of the human intellect started me many years ago on the track that led ultimately to this book, and to my late father, whose devotion to the written word has been a constant source of inspiration. I thank my children and grandchildren for their affection and cheerful acceptance of the demands this book has placed on my life. And, most of all, I am grateful to my husband, who for many years has encouraged me with his faith in the value of this project and supported me with his love and his wise advice.

Susanna Epp

CHAPTER 1

SPEAKING MATHEMATICALLY

Therefore O students study mathematics and do not build without foundations. —Leonardo da Vinci (1452–1519)

The aim of this book is to introduce you to a mathematical way of thinking that can serve you in a wide variety of situations. Often when you start work on a mathematical problem, you may have only a vague sense of how to proceed. You may begin by looking at examples, drawing pictures, playing around with notation, rereading the problem to focus on more of its details, and so forth. The closer you get to a solution, however, the more your thinking has to crystallize. And the more you need to understand, the more you need language that expresses mathematical ideas clearly, precisely, and unambiguously.

This chapter will introduce you to some of the special language that is a foundation for much mathematical thought, the language of variables, sets, relations, and functions. Think of the chapter like the exercises you would do before an important sporting event. Its goal is to warm up your mental muscles so that you can do your best.

1.1 Variables

A variable is sometimes thought of as a mathematical "John Doe" because you can use it as a placeholder when you want to talk about something but either (1) you imagine that it has one or more values but you don't know what they are, or (2) you want whatever you say about it to be equally true for all elements in a given set, and so you don't want to be restricted to considering only a particular, concrete value for it. To illustrate the first use, consider asking

> Is there a number with the following property: doubling it and adding 3 gives the same result as squaring it?

In this sentence you can introduce a variable to replace the potentially ambiguous word "it":

Is there a number x with the property that $2x + 3 = x^2$?

The advantage of using a variable is that it allows you to give a temporary name to what you are seeking so that you can perform concrete computations with it to help discover its possible values. To emphasize the role of the variable as a placeholder, you might write the following:

Is there a number $\square$ with the property that $2 \cdot \square + 3 = \square^2$?

The emptiness of the box can help you imagine filling it in with a variety of different values, some of which might make the two sides equal and others of which might not.