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Rainer Kress

# Linear Integral Equations

*Third Edition*

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*To the Memory of My Parents*

# Preface to the Third Edition

In the fourteen years since the second edition of this book appeared, linear integral equations have continued to be an active area of mathematics and they have revealed more of their power and beauty to me. Therefore I am pleased to have the opportunity to make adjustments and additions to the book's contents in this third edition. In the spirit of the two preceding editions, I have kept the balance between theory, applications and numerical methods. To preserve the character of an introduction as opposed to a research monograph, I have made no attempts to include most of the recent developments.

In addition to making corrections and additions throughout the text and updating the references, the following topics have been added. In order to make the introduction to the basic functional analytic tools more complete the Hahn–Banach extension theorem and the Banach open mapping theorem are now included in the text. The treatment of boundary value problems in potential theory has been extended by a more complete discussion of integral equations of the first kind in the classical Hölder space setting and of both integral equations of the first and second kind in the contemporary Sobolev space setting. In the numerical solution part of the book, I included a new collocation method for two-dimensional hypersingular boundary integral equations and the collocation method for the Lippmann–Schwinger equation based on fast Fourier transform techniques due to Vainikko. The final chapter of the book on inverse boundary value problems for the Laplace equation has been largely rewritten with special attention to the trilogy of decomposition, iterative and sampling methods.

Some of the additions to this third edition were written when I was visiting the Institut Mittag-Leffler, Djursholm, Sweden, in spring 2013 during the scientific program on *Inverse Problems and Applications*. I gratefully acknowledge the hospitality and the support.

Over the years most of the thirty-one PhD students that I supervised wrote their thesis on topics related to integral equations. Their work and my discussions with them have had significant influence on my own perspective on integral equations as presented in this book. Therefore, I take this opportunity to thank my PhD students as a group without listing them individually. A special note of thanks is given to my friend David Colton for reading over the new parts of the book and helping me with the English language.

I hope that this new edition of my book continues to attract readers to the field of integral equations and their applications.

Göttingen, Germany Rainer Kress

# Preface to the Second Edition

In the ten years since the first edition of this book appeared, integral equations and integral operators have revealed more of their mathematical beauty and power to me. Therefore, I am pleased to have the opportunity to share some of these new insights with the readers of this book. As in the first edition, the main motivation is to present the fundamental theory of integral equations, some of their main applications, and the basic concepts of their numerical solution in a single volume. This is done from my own perspective of integral equations; I have made no attempt to include all of the recent developments.

In addition to making corrections and adjustments throughout the text and updating the references, the following topics have been added: In Section 4.3 the presentation of the Fredholm alternative in dual systems has been slightly simplified and in Section 5.3 the short presentation on the index of operators has been extended. The treatment of boundary value problems in potential theory now includes proofs of the jump relations for single- and double-layer potentials in Section 6.3 and the solution of the Dirichlet problem for the exterior of an arc in two dimensions (Section 7.8). The numerical analysis of the boundary integral equations in Sobolev space settings has been extended for both integral equations of the first kind in Section 13.4 and integral equations of the second kind in Section 12.4. Furthermore, a short outline on fast $O(n \log n)$ solution methods has been added in Section 14.4. Because inverse obstacle scattering problems are now extensively discussed in the monograph [32], in the concluding Chapter 18 the application to inverse obstacle scattering problems has been replaced by an inverse boundary value problem for Laplace's equation.

I would like to thank Peter Hähner and Andreas Vogt for carefully reading the manuscript and for a number of suggestions for improving it. Thanks also go to those readers who helped me by letting me know the errors and misprints they found in the first edition.

I hope that this book continues to attract mathematicians and scientists to the field of integral equations and their applications.

Göttingen, Germany Rainer Kress

# Preface to the First Edition

I fell in love with integral equations about twenty years ago when I was working on my thesis, and I am still attracted by their mathematical beauty. This book will try to stimulate the reader to share this love with me.

Having taught integral equations a number of times I felt a lack of a text which adequately combines theory, applications and numerical methods. Therefore, in this book I intend to cover each of these fields with the same weight. The first part provides the basic Riesz–Fredholm theory for equations of the second kind with compact operators in dual systems including all functional analytic concepts necessary for developing this theory. The second part then illustrates the classical applications of integral equation methods to boundary value problems for the Laplace and the heat equation as one of the main historical sources for the development of integral equations, and also introduces Cauchy type singular integral equations. The third part is devoted to describing the fundamental ideas for the numerical solution of integral equations. Finally, in a fourth part, ill-posed integral equations of the first kind and their regularization are studied in a Hilbert space setting.

In order to make the book accessible not only to mathematicians but also to physicists and engineers I have planned it as self-contained as possible by requiring only a solid foundation in differential and integral calculus and, for parts of the book, in complex function theory. Some background in functional analysis will be helpful, but the basic concepts of the theory of normed spaces will be briefly reviewed, and all functional analytic tools which are relevant in the study of integral equations will be developed in the book. Of course, I expect the reader to be willing to accept the functional analytic language for describing the theory and the numerical solution of integral equations. I hope that I succeeded in finding the adequate compromise between presenting integral equations in the proper modern framework and the danger of being too abstract.

An introduction to integral equations cannot present a complete picture of all classical aspects of the theory and of all recent developments. In this sense, this book intends to tell the reader what I think appropriate to teach students in a two-semester course on integral equations. I am willing to admit that the choice of a few of the topics might be biased by my own preferences and that some important subjects are omitted.

I am indebted to Dipl.-Math. Peter Hähner for carefully reading the book, for checking the solutions to the problems and for a number of suggestions for valuable improvements. Thanks also go to Frau Petra Trapp who spent some time assisting me in the preparation of the LaTeX version of the text. And a particular note of thanks is given to my friend David Colton for reading over the book and helping me with the English language. Part of the book was written while I was on sabbatical leave at the Department of Mathematics at the University of Delaware. I gratefully acknowledge the hospitality.

Göttingen, Germany Rainer Kress

# Contents

# Chapter 1
# Introduction and Basic Functional Analysis

The topic of this book is linear integral equations of which

$$\int_a^b K(x, y)\varphi(y)\,dy = f(x), \quad x \in [a, b], \tag{1.1}$$

and

$$\varphi(x) - \int_a^b K(x, y)\varphi(y)\,dy = f(x), \quad x \in [a, b], \tag{1.2}$$

are typical examples. In these equations the function $\varphi$ is the unknown, and the so-called kernel $K$ and the right-hand side $f$ are given functions. Solving these integral equations amounts to determining a function $\varphi$ such that (1.1) or (1.2), respectively, are satisfied for all $a \le x \le b$. The term integral equation was first used by du Bois-Reymond [45] in 1888. The equations (1.1) and (1.2) carry the name of Fredholm because of his contributions to the field and are called *Fredholm integral equations* of the *first* and *second kind*, respectively. In the first equation the unknown function only occurs under the integral whereas in the second equation it also appears outside the integral. Later on we will see that this is more than just a formal difference between the two types of equations. A first impression on the difference can be obtained by considering the special case of a constant kernel $K(x, y) = c \neq 0$ for all $x, y \in [a, b]$. On one hand, it is easily seen that the equation of the second kind (1.2) has a unique solution given by

$$\varphi = f + \frac{c}{1 - c(b - a)} \int_a^b f(y)\,dy$$

if $c(b-a) \neq 1$. If $c(b-a) = 1$ then (1.2) is solvable if and only if $\int_a^b f(y)\,dy = 0$ and the general solution is given by $\varphi = f + \gamma$ with an arbitrary constant $\gamma$. On the other hand, the equation of the first kind (1.1) is solvable if and only if $f$ is a constant, $f(x) = \alpha$ for all $x \in [a, b]$ with an arbitrary constant $\alpha$. In this case every function $\varphi$ with $\int_a^b \varphi(y)\,dy = \alpha/c$ is a solution.

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Of course, the integration domains in (1.1) and (1.2) are not restricted to an interval $[a, b]$. In particular, the integration can be over multi-dimensional domains or surfaces and for the integral equation of the first kind the domain where the equation is required to be satisfied need not coincide with the integration domain.

We will regard the integral equations (1.1) and (1.2) as *operator equations*

$$A\varphi = f$$

and

$$\varphi - A\varphi = f$$

of the *first* and *second kind*, respectively, in appropriate normed function spaces.

The symbol $A : X \to Y$ will mean a single-valued mapping whose domain of definition is a set $X$ and whose range is contained in a set $Y$, i.e., for every $\varphi \in X$ the mapping $A$ assigns a unique element $A\varphi \in Y$. The *range* $A(X)$ is the set $A(X) := \{A\varphi : \varphi \in X\}$ of all image elements. We will use the terms *mapping, function,* and *operator* synonymously.

Existence and uniqueness of a solution to an operator equation can be equivalently expressed by the existence of the *inverse operator*. If for each $f \in A(X)$ there is only one element $\varphi \in X$ with $A\varphi = f$, then $A$ is said to be *injective* and to have an inverse $A^{-1} : A(X) \to X$ defined by $A^{-1}f := \varphi$. The inverse mapping has domain $A(X)$ and range $X$. It satisfies $A^{-1}A = I$ on $X$ and $AA^{-1} = I$ on $A(X)$, where $I$ denotes the identity operator mapping each element into itself. If $A(X) = Y$, then the mapping is said to be *surjective*. The mapping is called *bijective* if it is injective and surjective, i.e., if the inverse mapping $A^{-1} : Y \to X$ exists.

In the first part of the book we will present the Riesz–Fredholm theory for compact operators which, in particular, answers the question of existence and uniqueness of solutions to integral equations of the second kind with sufficiently smooth kernels. In order to develop the theory, we will assume that the reader is familiar with the elementary properties of linear spaces, normed spaces, and bounded linear operators. For convenience and to introduce notations, in this chapter, we briefly recall a few basic concepts from the theory of normed spaces, omitting most of the proofs. For a more detailed study, see Aubin [12], Brezis [21], Heuser [94], Kantorovic and Akilov [116], Rudin [209], and Taylor [229] among others.

## 1.1 Abel's Integral Equation

As an appetizer we consider Abel's integral equation that occurred as one of the first integral equations in mathematical history. A tautochrone is a planar curve for which the time taken by an object sliding without friction in uniform gravity to its lowest point is independent of its starting point. The problem to identify this curve was solved by Huygens in 1659 who, using geometrical tools, established that the tautochrone is a cycloid.

In 1823 Abel [1] attacked the more general problem of determining a planar curve such that the time of descent for a given starting height $y$ coincides with the value $f(y)$ of a given function $f$. The tautochrone then reduces to the special case when $f$ is a constant. Following Abel we describe the curve by $x = \psi(y)$ (with $\psi(0) = 0$) and, using the principle of conservation of energy, for the velocity $v$ at height $0 \le \eta \le y$ we obtain

$$\frac{1}{2}\, v^2 + g\eta = gy$$

where $g$ denotes the earth's gravity. Therefore, denoting arc length by $s$, the total time $f(y)$ required for the object to fall from $P = (\psi(y), y)$ to $P_0 = (0, 0)$ is given by

$$f(y) = \int_P^{P_0} \frac{ds}{v} = \int_0^y \sqrt{\frac{1 + [\psi'(\eta)]^2}{2g(y-\eta)}}\, d\eta.$$

Form this, setting

$$\varphi := \sqrt{\frac{1 + [\psi']^2}{2g}}$$

we obtain

$$f(y) = \int_0^y \frac{\varphi(y)}{\sqrt{y-\eta}}\, d\eta, \quad y > 0, \tag{1.3}$$

which is known as *Abel's integral equation*. Given the shape function $\varphi$, the falling time $f$ is obtained by simply evaluating the integral on the right-hand side of (1.3). Conversely, given the function $f$, finding $\varphi$ requires the solution of the integral equation (1.3) which certainly is a more challenging task.

For any solution $\varphi$ of (1.3) that is continuous on some interval $(0, a]$ and satisfies $|\varphi(y)| \le C/\sqrt{y}$ for all $y \in (0, a]$ and some constant $C$, by interchanging the order of integration, we obtain that

$$\begin{aligned}
\int_0^z \frac{f(y)}{\sqrt{z-y}}\, dy &= \int_0^z \frac{1}{\sqrt{z-y}} \int_0^y \frac{\varphi(\eta)}{\sqrt{y-\eta}}\, d\eta\, dy \\
&= \int_0^z \varphi(\eta) \int_\eta^z \frac{dy}{\sqrt{(z-y)(y-\eta)}}\, d\eta \\
&= \pi \int_0^z \varphi(\eta)\, d\eta.
\end{aligned}$$

Here, we substituted $y = z - (z-\eta)\cos^2 t$ with the result

$$\int_\eta^z \frac{dy}{\sqrt{(z-y)(y-\eta)}} = 2\int_0^{\pi/2} dt = \pi.$$

Therefore any solution of (1.3) has the form

$$\varphi(z) = \frac{1}{\pi} \frac{d}{dz} \int_0^z \frac{f(y)}{\sqrt{z-y}}\, dy, \quad z \in (0, a]. \tag{1.4}$$

Assuming that $f$ is continuously differentiable on $[0, a]$, by partial integration we obtain

$$\int_0^z \frac{f(y)}{\sqrt{z-y}}\, dy = 2\sqrt{z}\, f(0) + 2 \int_0^z \sqrt{z-y}\, f'(y)\, dy$$

and this transforms (1.4) into

$$\varphi(z) = \frac{1}{\pi} \left\{ \frac{f(0)}{\sqrt{z}} + \int_0^z \frac{f'(y)}{\sqrt{z-y}}\, dy \right\}, \quad z \in (0, a]. \tag{1.5}$$

Inserting (1.5) in (1.3) (after renaming the variables) and interchanging the order of integration as above shows that (1.4) indeed is a solution of (1.3).

For the special case of a constant $f = \pi\sqrt{a/2g}$ with $a > 0$ one obtains from (1.5) that

$$\varphi(y) = \sqrt{\frac{a}{2gy}}\,, \quad 0 < y \le a. \tag{1.6}$$

Note that the restriction $y \le a$ is a consequence of $2g\,\varphi^2 = 1 + [\psi']^2 \ge 1$. For the arc length $s$, we have

$$\frac{ds}{dy} = \sqrt{1 + [\psi']^2} = \sqrt{2g}\,\varphi$$

and from (1.6) it follows that

$$s(y) = 2\sqrt{ay}, \quad 0 \le y \le a.$$

For a convenient parameterization we set

$$y(t) = \frac{a}{2}\,(1 - \cos t), \quad 0 \le t \le \pi, \tag{1.7}$$

and obtain first

$$s(t) = 2a \sin\frac{t}{2}\,, \quad 0 \le t \le \pi,$$

and then with the aid of some trigonometric identities

$$x(t) = \frac{a}{2}\,(t + \sin t), \quad 0 \le t \le \pi. \tag{1.8}$$

Hence, the tautochrone as given by the parameterization (1.7) and (1.8) is the cycloid generated as the trajectory described by a point on the circle of radius $a/2$ when the circle is rolling along the straight line $y = a$. This property of the cycloid, together with the fact that the involute of a cycloid again is a cycloid, was exploited by Huygens to build a cycloidal pendulum for which the frequency of the oscillations does not depend on the amplitude in contrast to the circular pendulum.

## 1.2 Convergence and Continuity

**Definition 1.1.** Let $X$ be a complex (or real) linear space (vector space). A function $\|\cdot\| : X \to \mathbb{R}$ with the properties

(N1) $\|\varphi\| \geq 0,$ (positivity)

(N2) $\|\varphi\| = 0$ if and only if $\varphi = 0,$ (definiteness)

(N3) $\|\alpha\varphi\| = |\alpha|\,\|\varphi\|,$ (homogeneity)

(N4) $\|\varphi + \psi\| \leq \|\varphi\| + \|\psi\|,$ (triangle inequality)

for all $\varphi, \psi \in X$, and all $\alpha \in \mathbb{C}$ (or $\mathbb{R}$) is called a *norm* on $X$. A linear space $X$ equipped with a norm is called a *normed space*.

As a consequence of (N3) and (N4) we note the second triangle inequality

$$|\,\|\varphi\| - \|\psi\|\,| \leq \|\varphi - \psi\|. \tag{1.9}$$

For two elements in a normed space $\|\varphi - \psi\|$ is called the *distance* between $\varphi$ and $\psi$.

**Definition 1.2.** A sequence $(\varphi_n)$ of elements of a normed space $X$ is called *convergent* if there exists an element $\varphi \in X$ such that $\lim_{n\to\infty} \|\varphi_n - \varphi\| = 0$, i.e., if for every $\varepsilon > 0$ there exists an integer $N(\varepsilon)$ such that $\|\varphi_n - \varphi\| < \varepsilon$ for all $n \geq N(\varepsilon)$. The element $\varphi$ is called the *limit* of the sequence $(\varphi_n)$, and we write

$$\lim_{n\to\infty} \varphi_n = \varphi \quad \text{or} \quad \varphi_n \to \varphi,\ n \to \infty.$$

Note that by (N4) the limit of a convergent sequence is uniquely determined. A sequence that does not converge is called *divergent*.

**Definition 1.3.** A function $A : U \subset X \to Y$ mapping a subset $U$ of a normed space $X$ into a normed space $Y$ is called *continuous* at $\varphi \in U$ if $\lim_{n\to\infty} A\varphi_n = A\varphi$ for every sequence $(\varphi_n)$ from $U$ with $\lim_{n\to\infty} \varphi_n = \varphi$. The function $A : U \subset X \to Y$ is called continuous if it is continuous for all $\varphi \in U$.

An equivalent definition is the following: A function $A : U \subset X \to Y$ is continuous at $\varphi \in U$ if for every $\varepsilon > 0$ there exists $\delta > 0$ such that $\|A\varphi - A\psi\| < \varepsilon$ for all $\psi \in U$ with $\|\varphi - \psi\| < \delta$. Here we have used the same symbol $\|\cdot\|$ for the norms on $X$ and $Y$. The function $A$ is called *uniformly continuous* if for every $\varepsilon > 0$ there exists $\delta > 0$ such that $\|A\varphi - A\psi\| < \varepsilon$ for all $\varphi, \psi \in U$ with $\|\varphi - \psi\| < \delta$.

Note that by (1.9) the norm is a continuous function.

In our study of integral equations the basic example of a normed space will be the linear space $C[a, b]$ of continuous real- or complex-valued functions $\varphi$ defined on an interval $[a, b] \subset \mathbb{R}$ furnished either with the *maximum norm*

$$\|\varphi\|_\infty := \max_{x\in[a,b]} |\varphi(x)|$$

or the *mean square norm*

$$\|\varphi\|_2 := \left( \int_a^b |\varphi(x)|^2 dx \right)^{1/2}.$$

Convergence of a sequence of continuous functions in the maximum norm is equivalent to uniform convergence, and convergence in the mean square norm is called *mean square convergence*. Throughout this book, unless stated otherwise, we always assume that $C[a,b]$ (or $C(G)$, i.e., the space of continuous real- or complex-valued functions on compact subsets $G \subset \mathbb{R}^m$) is equipped with the maximum norm.

**Definition 1.4.** Two norms on a linear space are called *equivalent* if they have the same convergent sequences.

**Theorem 1.5.** *Two norms $\|\cdot\|_a$ and $\|\cdot\|_b$ on a linear space $X$ are equivalent if and only if there exist positive numbers $c$ and $C$ such that*

$$c\|\varphi\|_a \le \|\varphi\|_b \le C\|\varphi\|_a$$

*for all $\varphi \in X$. The limits with respect to the two norms coincide.*

*Proof.* Provided that the conditions are satisfied, from $\|\varphi_n - \varphi\|_a \to 0$, $n \to \infty$, it follows $\|\varphi_n - \varphi\|_b \to 0$, $n \to \infty$, and vice versa.

Conversely, let the two norms be equivalent and assume that there is no $C > 0$ such that $\|\varphi\|_b \le C\|\varphi\|_a$ for all $\varphi \in X$. Then there exists a sequence $(\varphi_n)$ satisfying $\|\varphi_n\|_a = 1$ and $\|\varphi_n\|_b \ge n^2$. Now, the sequence $(\psi_n)$ with $\psi_n := n^{-1}\varphi_n$ converges to zero with respect to $\|\cdot\|_a$, whereas with respect to $\|\cdot\|_b$ it is divergent because of $\|\psi_n\|_b \ge n$. □

**Theorem 1.6.** *On a finite-dimensional linear space all norms are equivalent.*

*Proof.* In a linear space $X$ with finite dimension $m$ and basis $f_1, \dots, f_m$ every element can be expressed in the form

$$\varphi = \sum_{k=1}^{m} \alpha_k f_k.$$

As is easily verified,

$$\|\varphi\|_\infty := \max_{k=1,\dots,m} |\alpha_k| \tag{1.10}$$

defines a norm on $X$. Let $\|\cdot\|$ denote any other norm on $X$. Then, by the triangle inequality we have

$$\|\varphi\| \le C\|\varphi\|_\infty$$

for all $\varphi \in X$, where

$$C := \sum_{k=1}^{m} \|f_k\|.$$

Assume that there is no $c > 0$ such that $c\|\varphi\|_\infty \le \|\varphi\|$ for all $\varphi \in X$. Then there exists a sequence $(\varphi_n)$ with $\|\varphi_n\| = 1$ such that $\|\varphi_n\|_\infty \ge n$. Consider the sequence

$(\psi_n)$ with $\psi_n := \|\varphi_n\|_\infty^{-1}\varphi_n$ and write

$$\psi_n = \sum_{k=1}^{m} \alpha_{kn} f_k.$$

Because of $\|\psi_n\|_\infty = 1$ each of the sequences $(\alpha_{kn})$, $k = 1, \dots, m$, is bounded in $\mathbb{C}$. Hence, by the Bolzano–Weierstrass theorem we can select convergent subsequences $\alpha_{k,n(j)} \to \alpha_k$, $j \to \infty$, for each $k = 1, \dots, m$. This now implies $\|\psi_{n(j)} - \psi\|_\infty \to 0$, $j \to \infty$, where

$$\psi := \sum_{k=1}^{m} \alpha_k f_k,$$

and $\|\psi_{n(j)} - \psi\| \le C\|\psi_{n(j)} - \psi\|_\infty \to 0$, $j \to \infty$. But on the other hand we have $\|\psi_n\| = 1/\|\varphi_n\|_\infty \to 0$, $n \to \infty$. Therefore, $\psi = 0$, and consequently $\|\psi_{n(j)}\|_\infty \to 0$, $j \to \infty$, which contradicts $\|\psi_n\|_\infty = 1$ for all $n$. □

For an element $\varphi$ of a normed space $X$ and a positive number $r$ the set $B(\varphi; r) := \{\psi \in X : \|\psi - \varphi\| < r\}$ is called the *open ball* of radius $r$ and center $\varphi$, the set $B[\varphi; r] := \{\psi \in X : \|\psi - \varphi\| \le r\}$ is called a *closed ball*.

**Definition 1.7.** A subset $U$ of a normed space $X$ is called *open* if for each element $\varphi \in U$ there exists $r > 0$ such that $B(\varphi; r) \subset U$.

Obviously, open balls are open.

**Definition 1.8.** A subset $U$ of a normed space $X$ is called *closed* if it contains all limits of convergent sequences of $U$.

A subset $U$ of a normed space $X$ is closed if and only if its complement $X \setminus U$ is open. Obviously, closed balls are closed. In particular, using the norm (1.10), it can be seen that finite-dimensional subspaces of a normed space are closed.

**Definition 1.9.** The *closure* $\overline{U}$ of a subset $U$ of a normed space $X$ is the set of all limits of convergent sequences of $U$. A set $U$ is called *dense* in another set $V$ if $V \subset \overline{U}$, i.e., if each element in $V$ is the limit of a convergent sequence from $U$.

A subset $U$ is closed if and only if it coincides with its closure. By the Weierstrass approximation theorem (see [40]) the linear subspace $P$ of polynomials is dense in $C[a, b]$ with respect to the maximum norm and the mean square norm.

**Definition 1.10.** A subset $U$ of a normed space $X$ is called *bounded* if there exists a positive number $C$ such that $\|\varphi\| \le C$ for all $\varphi \in U$.

Convergent sequences are bounded.