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Linear Systems

A Measurement Based Approach

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To Gisele, for her love and support

SPB

To My Wife, Kuisook

LHK

To my mother, Shamsozzoha,
my father, Mohammad Farid,
and my brother, Mehrdad

DNM

Preface

This monograph presents the recent results obtained by us on the analysis, synthesis and design of systems described by linear equations. As is well known, linear equations arise in most branches of science and engineering as well as social, biological and economic systems. The novelty of our approach lies in the fact that no models of the system are assumed to be available, nor are they required. Instead, we show that a few measurements made on the system can be processed strategically to directly extract design values that meet specifications without constructing a model of the system, implicitly or explicitly. We illustrate these new concepts by applying them to linear D.C. and A.C. circuits, mechanical, civil and hydraulic systems, signal flow block diagrams and control systems. These applications are preliminary and suggest many open problems. We acknowledge many productive discussions with our colleagues A. Datta, Hazem Nounou, Mohamed Nounou and our graduate students Ritwik Layek and Sirisha Kallakuri.

Earlier research by us has shown that the representation of complex systems by high order models with many parameters may lead to fragility, that is, the drastic change of system behaviour under infinitesimally small perturbations of these parameters. This led to research on model-free measurement-based approaches to design. The results presented in this monograph are our latest effort in this direction and we hope they will lead to attractive alternatives to model-based design of engineering and other systems. We also anticipate applications to robust, adaptive and fault tolerant control.

College Station, USA, June 25, 2013

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Chapter 1
Linear Equations with Parameters

In this chapter, we describe some basic results on the solution of linear equations containing parameters, and the nature of the parameterized solutions. We describe how measurements can be used to extract these parameterized solutions when the equations or models are unknown. These are presented as a generalized Superposition Theorem and a Measurement Theorem.

1.1 Introduction

Consider the system of linear equations

$$\mathbf{A}\mathbf{x} = \mathbf{b}, \tag{1.1}$$

where $\mathbf{A}$ is an $n \times n$ matrix, and $\mathbf{x}$ and $\mathbf{b}$ are $n \times 1$ vectors all with real or complex entries. Let $|.|$ denotes the determinant. Assuming that $|\mathbf{A}| \neq 0$, there exists a unique solution $\mathbf{x}$ and, by Cramer's rule, the ith component x_i of $\mathbf{x}$ is given by

$$x_i = \frac{\left|\mathbf{A}^i(\mathbf{b})\right|}{|\mathbf{A}|}, \qquad i = 1, 2, \ldots, n \tag{1.2}$$

where $\mathbf{A}^i(\mathbf{b})$ is the matrix obtained by replacing the ith column of $\mathbf{A}$ by $\mathbf{b}$.

In many physical problems, $\mathbf{A}$ and $\mathbf{b}$ contain parameters that need to be chosen or designed, as illustrated in the example below.

Example 1.1. Consider the circuit shown in Fig. 1.1. V is the ideal voltage source, I is the ideal current source, R_1, R_2, R_3 are linear resistors, and R_4 is a gyrator resistance. The gyrator is a linear two port device where the instantaneous currents and the instantaneous voltages are related by $V_2 = R_4 I_2$ and $V_1 = -R_4 I_3$. V_{amp} is the dependent voltage of the amplifier where $V_{\text{amp}} = K I_1$, and K is the amplifier gain. The equations of the system can be written in the following matrix form by applying Kirchhoff's current and voltage laws,

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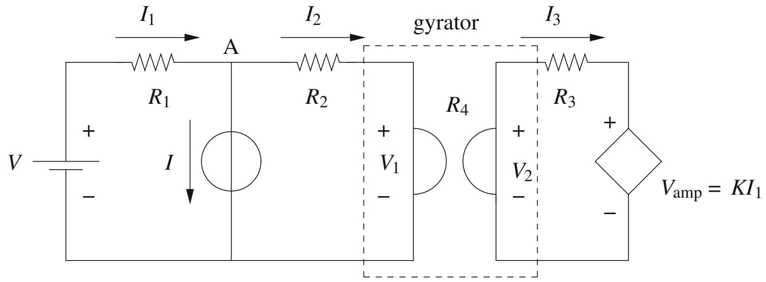

Fig. 1.1 A general circuit

$$\underbrace{\begin{bmatrix} 1 & -1 & 0 \\ R_1 & R_2 & -R_4 \\ K & -R_4 & R_3 \end{bmatrix}}_{\mathbf{A}} \underbrace{\begin{bmatrix} I_1 \\ I_2 \\ I_3 \end{bmatrix}}_{\mathbf{x}} = \underbrace{\begin{bmatrix} I \\ V \\ 0 \end{bmatrix}}_{\mathbf{b}}. \tag{1.3}$$

To fix notation, we introduce the *parameter* vector $\mathbf{p}$ and the vector of *sources* $\mathbf{q}$:

$$\mathbf{p} := \begin{bmatrix} R_1 \\ R_2 \\ R_3 \\ R_4 \\ K \end{bmatrix} = \begin{bmatrix} p_1 \\ p_2 \\ p_3 \\ p_4 \\ p_5 \end{bmatrix} \text{ and } \mathbf{q} := \begin{bmatrix} I \\ V \end{bmatrix} = \begin{bmatrix} q_1 \\ q_2 \end{bmatrix}, \tag{1.4}$$

so that (1.1) can be rewritten showing explicitly the dependence on the parameter vector $\mathbf{p}$ and the source vector $\mathbf{q}$ as

$$\mathbf{A}(\mathbf{p})\mathbf{x} = \mathbf{b}(\mathbf{q}). \tag{1.5}$$

Thus, (1.2) can also be rewritten explicitly showing the parameterized solution as

$$x_i(\mathbf{p}, \mathbf{q}) = \frac{\left|\mathbf{A}^i(\mathbf{p}, \mathbf{b}(\mathbf{q}))\right|}{|\mathbf{A}(\mathbf{p})|} := \frac{|\mathbf{B}_i(\mathbf{p}, \mathbf{q})|}{|\mathbf{A}(\mathbf{p})|}, \quad i = 1, 2, \ldots, n. \tag{1.6}$$

More generally, if

$$y(\mathbf{p}, \mathbf{q}) = \mathbf{c}^T \mathbf{x}(\mathbf{p}, \mathbf{q}) = c_1 x_1(\mathbf{p}, \mathbf{q}) + \cdots + c_n x_n(\mathbf{p}, \mathbf{q}) \tag{1.7}$$

is an output of interest, it follows that

$$y(\mathbf{p}, \mathbf{q}) = \sum_{i=1}^{n} c_i \left(\frac{|\mathbf{B}_i(\mathbf{p}, \mathbf{q})|}{|\mathbf{A}(\mathbf{p})|} \right). \tag{1.8}$$

1.2 Parameterized Solutions

Motivated by the above example we consider henceforth the general representation of an arbitrary linear system to be given by (1.5), (1.6) and (1.8). To develop the formula (1.6) in more detail, we note that in (1.3) the parameter $\mathbf{p}$ appears *affinely* in $\mathbf{A}(\mathbf{p})$. Thus, we can write

$$\mathbf{A}(\mathbf{p}) = \mathbf{A}_0 + p_1\mathbf{A}_1 + p_2\mathbf{A}_2 + \cdots + p_l\mathbf{A}_l. \tag{1.9}$$

To proceed, consider the special case of a scalar parameter $\mathbf{p} = p_1$ and

$$\mathbf{A}(\mathbf{p}) = \mathbf{A}_0 + p_1\mathbf{A}_1. \tag{1.10}$$

Lemma 1.1. *With $\mathbf{A}(\mathbf{p})$ as in (1.10), $|\mathbf{A}(\mathbf{p})|$ is a polynomial of degree at most r_1 in p_1 where*

$$r_1 = \text{rank}\,[\mathbf{A}_1]\,. \tag{1.11}$$

Proof. The proof follows easily from the properties of determinants. □

Example 1.2. Consider the following 2×2 matrix $\mathbf{A}$

$$\mathbf{A}(\mathbf{p}) = \begin{bmatrix} 1+p & 1-p \\ p & 2+p \end{bmatrix}, \tag{1.12}$$

which can be written as

$$\mathbf{A}(\mathbf{p}) = \underbrace{\begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}}_{\mathbf{A}_0} + p \underbrace{\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}}_{\mathbf{A}_1}. \tag{1.13}$$

Matrix $\mathbf{A}_1$ has rank 2, and we say that matrix $\mathbf{A}(\mathbf{p})$ is of rank 2 with respect to p. Therefore, $|\mathbf{A}(\mathbf{p})|$ will be a polynomial, in p, of degree at most 2, by Lemma 1.1. Calculating $|\mathbf{A}(\mathbf{p})|$ yields

$$|\mathbf{A}(\mathbf{p})| = 2p^2 + 2p + 2. \tag{1.14}$$

Example 1.3. Consider

$$\mathbf{A}(\mathbf{p}) = \begin{bmatrix} 2+p & 1.5+p \\ 1.5+p & 1+p \end{bmatrix} = \underbrace{\begin{bmatrix} 2 & 1.5 \\ 1.5 & 1 \end{bmatrix}}_{\mathbf{A}_0} + p \underbrace{\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}}_{\mathbf{A}_1}, \tag{1.15}$$

so that $\text{rank}[\mathbf{A}_1] = 1$. Here $r_1 = 1$, but

$$|\mathbf{A}(\mathbf{p})| = 0.75, \tag{1.16}$$

is a polynomial of degree 0 in p.

Lemma 1.2. *With* $\mathbf{A}(\mathbf{p})$ *as in* (1.9), *let*

$$r_i = \text{rank}\,[\mathbf{A}_i]\,, \quad i = 1, 2, \ldots, l. \tag{1.17}$$

Then, $|\mathbf{A}(\mathbf{p})|$ *is a multivariate polynomial in* $\mathbf{p}$ *of degree* r_i *or less in* p_i, $i = 1, 2, \ldots, l$ *and*

$$|\mathbf{A}(\mathbf{p})| = \sum_{i_l=0}^{r_l} \cdots \sum_{i_2=0}^{r_2} \sum_{i_1=0}^{r_1} \alpha_{i_1 i_2 \cdots i_l} p_1^{i_1} p_2^{i_2} \cdots p_l^{i_l} := \alpha(\mathbf{p}). \tag{1.18}$$

Also, if the parameter $\mathbf{q}$ *is fixed, say* $\mathbf{q} = \mathbf{q}_0$, *then*

$$|\mathbf{B}_i(\mathbf{p}, \mathbf{q}_0)| = \sum_{i_l=0}^{t_l} \cdots \sum_{i_2=0}^{t_2} \sum_{i_1=0}^{t_1} \beta_{i_1 i_2 \cdots i_l} p_1^{i_1} p_2^{i_2} \cdots p_l^{i_l} := \beta_i(\mathbf{p}, \mathbf{q}_0), \tag{1.19}$$

where $\mathbf{B}_i(\mathbf{p}, \mathbf{q}_0)$ *is the matrix obtained by replacing the* ith *column of* $\mathbf{A}(\mathbf{p})$, *in* (1.5), *by the vector* $\mathbf{b}(\mathbf{q}_0)$, *and*

$$t_i = \text{rank}\,[\mathbf{B}_i] \leq r_i\,, \quad i = 1, 2, \ldots, l. \tag{1.20}$$

Proof. This follows immediately from Lemma 1.1. □

Remark 1.1. In the formula (1.18), the number of coefficients $\alpha_{i_1 i_2 \cdots i_l}$ are $\prod_{i=1}^{l}(r_i + 1)$.

Based on the above formula, we have the following characterization of parameterized solutions.

Theorem 1.1. *With* $\mathbf{A}(\mathbf{p})$ *and* $\mathbf{b}(\mathbf{q})$ *as in* (1.5) *and* (1.9),

$$x_i(\mathbf{p}, \mathbf{q}_0) = \frac{\beta_i(\mathbf{p}, \mathbf{q}_0)}{\alpha(\mathbf{p})}, \quad i = 1, 2, \ldots, n, \tag{1.21}$$

where $\beta_i(\mathbf{p}, \mathbf{q}_0)$, $i = 1, 2, \ldots, n$ *and* $\alpha(\mathbf{p})$ *are multivariate polynomials in* $\mathbf{p}$.

Proof. The proof follows from (1.6) and Lemma 1.2. □

1.3 Measurements and Models

1.3.1 Polynomial Models

In this section we introduce some simple ideas related to measurements and models. Suppose that the true equation of a system is as follows:

$$y = c_0 + c_1 x + \cdots + c_n x^n, \tag{1.22}$$

where x is an "input" and y is an "output". Now, assume that the mathematical representation (1.22) is not yet known and we are interested in finding a polynomial model of the system by means of measurements. In (1.22) we suppose that the order n and the coefficients c_i, $i = 0, 1, \ldots, n$ are unknown. Consider a polynomial model of the system, of the type:

$$y^{model} = \alpha_0 + \alpha_1 x + \cdots + \alpha_m x^m, \tag{1.23}$$

where x is an "input" and y is an "output" and $\alpha_0, \alpha_1, \ldots, \alpha_m$ are unknown. Suppose experiments are made on the real system by inputting different values $x_1, x_2, \ldots, x_{m+1}$ and measuring the resulting outputs $y_1, y_2, \ldots, y_{m+1}$. The measurement matrix equation is

$$\underbrace{\begin{bmatrix} 1 & x_1 & x_1^2 & \cdots & x_1^m \\ 1 & x_2 & x_2^2 & \cdots & x_2^m \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & x_{m+1} & x_{m+1}^2 & \cdots & x_{m+1}^m \end{bmatrix}}_{\mathbf{M}_m} \underbrace{\begin{bmatrix} \alpha_0 \\ \alpha_1 \\ \vdots \\ \alpha_m \end{bmatrix}}_{\alpha} = \underbrace{\begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_{m+1} \end{bmatrix}}_{\mathbf{y}}, \tag{1.24}$$

where $|\mathbf{M}_m| = (x_1 - x_2)(x_2 - x_3)\cdots(x_m - x_{m+1})$ and since the input values $x_1, x_2, \ldots, x_{m+1}$ are different, then $|\mathbf{M}_m| \neq 0$. Therefore, (1.24) can be uniquely solved for the vector $\alpha = [\alpha_0, \alpha_1, \ldots, \alpha_m]^T$, and the candidate model (1.23) can be determined. The following three cases are possible:

Case 1: $m = n$.
In this case $\alpha_m \neq 0$, and if another experiment is performed by inputting a different x, called x_{m+2}, and measuring y_{m+2}. Then it must be true that

$$y_{m+2} = \alpha_0 + \alpha_1 x_{m+2} + \cdots + \alpha_m x_{m+2}^m, \tag{1.25}$$

where $\alpha = [\alpha_0, \alpha_1, \ldots, \alpha_m]^T$ is determined from (1.24).

Case 2: $m < n$.
In this situation if another experiment is performed by inputting a different x, called x_{m+2}, then

$$y_{m+2} \neq \alpha_0 + \alpha_1 x_{m+2} + \cdots + \alpha_m x_{m+2}^m. \tag{1.26}$$

In such a case the order of the polynomial model has to be increased.

Case 3: $m > n$.
In this case $\alpha_m = 0$ (and possibly $\alpha_{m-1} = 0, \ldots$). This fact can be detected from the matrix

$$\mathbf{M} = \begin{bmatrix} 1 & x_1 & x_1^2 & \cdots & y_1 \\ 1 & x_2 & x_2^2 & \cdots & y_2 \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & x_{m+1} & x_{m+1}^2 & \cdots & y_{m+1} \end{bmatrix}, \tag{1.27}$$

which must have rank less than $m + 1$. Then one can reduce the order and test the rank of the corresponding matrix again until a full rank matrix is obtained. If one performs another experiment on the system, this model should predict the output correctly; otherwise, we are in Case 2.

Example 1.4. This example illustrates how a polynomial can be constructed (through measurements) to model an actual system. Suppose that the actual system is

$$y = 2 - x + x^3, \tag{1.28}$$

and we begin with the following polynomial model

$$y^{model} = \alpha_0 + \alpha_1 x, \tag{1.29}$$

where α_0 and α_1 are constants to be determined. In order to determine α_0 and α_1, one may perform two experiments by inputting two different values for x into the actual system and measuring the output $y^{measured}$ which is (assumed to be) equal to y. Let us for example set $x_1 = 1,\ x_2 = 2$ and measure $y_1 = 2,\ y_2 = 8$. Thus, we have

$$\underbrace{\begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}}_{\mathbf{M}_2} \underbrace{\begin{bmatrix} \alpha_0 \\ \alpha_1 \end{bmatrix}}_{\alpha} = \underbrace{\begin{bmatrix} 2 \\ 8 \end{bmatrix}}_{\mathbf{y}}, \tag{1.30}$$

which can be solved for $\alpha_0 = -4$ and $\alpha_1 = 6$. Hence,

$$y^{model} = -4 + 6x. \tag{1.31}$$

Suppose we do another experiment using $x_3 = 5$ and measure $y_3 = 122$, but

$$y^{model}|_{x=5} = 26 \neq 122. \tag{1.32}$$

This implies that the order of the polynomial model has to be increased. Now, let us propose

$$y^{model} = \alpha_0 + \alpha_1 x + \alpha_2 x^2, \tag{1.33}$$

and carry out the experiments using $x_1 = 1,\ x_2 = 2,\ x_3 = 5$ which yields $y_1 = 2,\ y_2 = 8,\ y_3 = 122$. Then, we have

$$\underbrace{\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 5 & 25 \end{bmatrix}}_{\mathbf{M}_3} \underbrace{\begin{bmatrix} \alpha_0 \\ \alpha_1 \\ \alpha_2 \end{bmatrix}}_{\alpha} = \underbrace{\begin{bmatrix} 2 \\ 8 \\ 122 \end{bmatrix}}_{\mathbf{y}}, \tag{1.34}$$

which results in

$$y^{model} = 12 - 18x + 8x^2. \tag{1.35}$$

Assume we perform another experiment by setting $x_4 = 7$ and measure $y_4 = 338$, but

$$y^{model}|_{x=7} = 278 \neq 338. \tag{1.36}$$

This implies that the order of the polynomial model has to be increased; thus, we consider

$$y^{model} = \alpha_0 + \alpha_1 x + \alpha_2 x^2 + \alpha_3 x^3. \tag{1.37}$$

To determine the values of the constants $\alpha_0, \alpha_1, \alpha_2$ and α_3, let us set $x_1 = 1,\ x_2 = 2,\ x_3 = 5,\ x_4 = 7$ and measure $y_1 = 2,\ y_2 = 8,\ y_3 = 122,\ y_4 = 338$. The following set of measurement equations can be formed:

$$\underbrace{\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & 8 \\ 1 & 5 & 25 & 125 \\ 1 & 7 & 49 & 343 \end{bmatrix}}_{\mathbf{M}_4} \underbrace{\begin{bmatrix} \alpha_0 \\ \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix}}_{\alpha} = \underbrace{\begin{bmatrix} 2 \\ 8 \\ 122 \\ 338 \end{bmatrix}}_{\mathbf{y}}, \tag{1.38}$$

and uniquely solved for the vector of unknown constants $\alpha = [\alpha_0, \alpha_1, \alpha_2, \alpha_3]^T$, which gives $\alpha = [2, -1, 0, 1]^T$ so that

$$y^{model} = 2 - x + x^3. \tag{1.39}$$

Let us perform another experiment using $x_5 = 9$ and measure $y_5 = 722$. It can be verified that

$$y^{model}|_{x=9} = y_5 = 722. \tag{1.40}$$

Therefore, (1.39) is the correct model.

1.3.2 Rational Models

Here, we extend the idea presented in Sect. 1.3.1 to rational models. Suppose that the mathematical representation of a system is:

$$y = \frac{c_0 + c_1 x + \cdots + c_{m-1} x^{m-1} + c_m x^m}{d_0 + d_1 x + \cdots + d_{n-1} x^{n-1} + x^n}. \tag{1.41}$$

Consider the rational model

$$y^{model} = \frac{\alpha_0 + \alpha_1 x + \cdots + \alpha_s x^s}{\beta_0 + \beta_1 x + \cdots + x^r}. \tag{1.42}$$

Let $k := s + r + 1$, denote the number of unknowns. Similar to the previous case, suppose that experiments are conducted by inputting different values $x_1, x_2, \ldots, x_k$ and measuring the resulting outputs $y_1, y_2, \ldots, y_k$. The measurement matrix equation may be written as

$$\underbrace{\begin{bmatrix} 1 & x_1 & \cdots & x_1^s & -y_1 & -y_1 x_1 & \cdots & -y_1 x_1^{r-1} \\ 1 & x_2 & \cdots & x_2^s & -y_2 & -y_2 x_2 & \cdots & -y_2 x_2^{r-1} \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ 1 & x_k & \cdots & x_k^s & -y_k & -y_k x_k & \cdots & -y_k x_k^{r-1} \end{bmatrix}}_{\mathbf{M}_k} \underbrace{\begin{bmatrix} \alpha_0 \\ \alpha_1 \\ \vdots \\ \alpha_s \\ \beta_0 \\ \beta_1 \\ \vdots \\ \beta_{r-1} \end{bmatrix}}_{\gamma} = \underbrace{\begin{bmatrix} y_1 x_1^r \\ y_2 x_2^r \\ \vdots \\ y_k x_k^r \end{bmatrix}}_{\mathbf{y}}, \tag{1.43}$$

and can be uniquely solved for the unknowns $\gamma = [\alpha_0, \ldots, \alpha_s, \beta_0, \ldots, \beta_{r-1}]^T$, if and only if $|\mathbf{M}_k| \neq 0$. In a problem of this type, the behavior of the system at large values of the input determines whether the system rational function is proper ($m \leq n$), strictly proper ($m < n$) or improper ($m > n$). Therefore, we consider the following two cases:

Case A: $m \leq n$.
In such a case, if possible, one may conduct at least two experiments for sufficiently large values of the input x. If the output measurements, called y_1 and y_2, are (approximately) equal and $y_1 \approx y_2 \not\approx 0$, then $m = n$, and if $y_1 \approx y_2 \approx 0$, then $m < n$. We can write the following

$$y^{model} = y_\infty + y^{strictly\ proper\ model}, \tag{1.44}$$

where y_∞ is the measured value of the system output for a sufficiently large value of the input x, in this case.

Case B: $m > n$.
One can determine if the system rational function is improper by conducting at least two experiments for sufficiently large values of the input x. If the output measurements are not (approximately) equal then $m > n$. In this case the behavior of the system at large values of the input can be modeled as a polynomial function $y^{polynomial\ model}$, using the method presented in Sect. 1.3.1. Hence, we have

$$y^{model} = y^{polynomial\ model} + y^{strictly\ proper\ model}. \tag{1.45}$$

With $y^{polynomial\ model}$ or y_∞ in hand, the problem reduces to determining a strictly proper rational model $y^{strictly\ proper\ model}$. One may consider a model as in (1.42) with $s = r - 1$:

$$y^{strictly\ proper\ model} = \frac{\alpha_0 + \alpha_1 x + \cdots + \alpha_{r-1} x^{r-1}}{\beta_0 + \beta_1 x + \cdots + \beta_r x^r}, \tag{1.46}$$

and determine the coefficients α's and β's by conducting a sufficient number of experiments. As in Sect. 1.3.1, three cases are possible:

Case 1: $r = n$.
In this case $\beta_r \neq 0$, and if another experiment is performed by inputting a different x, say x_{2r+2}, then $y^{model}|_{x_{2r+2}} = y_{2r+2}$.

Case 2: $r < n$.
Here, if one conducts another experience using a different input, for example x_{2r+2}, then $y^{model}|_{x_{2r+2}} \neq y_{2r+2}$. Thus, the order of the numerator and denominator polynomials in (1.46) has to be increased simultaneously by the same value (keeping the relative degree to be 1).

Case 3: $r > n$.
In this situation $\beta_r = 0$ (and possibly $\beta_{r-1} = 0, \ldots$); hence, the order of the denominator and numerator polynomials have to be reduced such that the coefficient of the highest order term in the denominator and numerator are nonzero. If another experiment is done using a different x, called x_{2r+2}, then the predicted value $y^{model}|_{x_{2r+2}}$ is equal to the measured output y_{2r+2}.

Example 1.5. Suppose that the mathematical representation of a system is

$$y = \frac{10}{3 + x^2}. \tag{1.47}$$

Let us, first, examine the behavior of the system at large values of input by inputting $x_1 = 1,000,\ x_2 = 1,200$. We obtain $y_1 \approx y_2 \approx 0$, and thus conclude that $m < n$. Now, consider the following model

$$y^{model} = \frac{\alpha_0}{\beta_0 + x}. \tag{1.48}$$

Let us input two different values for x, for example $x_1 = 1,\ x_2 = 2$, into the system and measure the output y, which yields $y_1 = 2.5,\ y_2 = 1.4286$. Thus, we obtain

$$y = \frac{3.333}{0.333 + x}. \tag{1.49}$$

If we perform another experiment by inputting $x_3 = 4$, we get $y_3 = 0.5263$, but for which $y^{model}|_{x=4} = 0.7692 \neq 0.5263$. Hence, in the next step, we increase the order of the polynomials in the numerator and denominator of (1.48) by 1:

$$y^{model} = \frac{\alpha_0 + \alpha_1 x}{\beta_0 + \beta_1 x + x^2}. \tag{1.50}$$

Table 1.1 summarized the numerical values of the experiments performed for this model.

One can determine the coefficients and obtain

$$y^{model} = \frac{10}{3 + x^2}. \tag{1.51}$$

Now, suppose that one performs another experiment using $x_5 = 10$ and obtains $y_5 = 0.0971$. This can be predicted by substituting $x = 10$ into (1.51) which results in $y^{model}|_{x=10} = y_5 = 0.0971$.

1.4 Determining a General Parameterized Solution from Measurements

In many synthesis and design problems a model is unavailable, but it is necessary to know the behavior of system variable with respect to various parameters. In this

Table 1.1 Numerical values of the experiments performed for example 1.5

Input value	Measured output
1	2.5
2	1.4286
4	0.5263
7	0.1923

section we develop a measurement based approach to the determination of the solution function of a linear system of equations when the model is not known. Specifically, we consider the linear system or model

$$\mathbf{A}(\mathbf{p})\mathbf{x} = \mathbf{b}(\mathbf{q}), \tag{1.52}$$

where the matrix $\mathbf{A}(\mathbf{p})$ and vector $\mathbf{b}(\mathbf{q})$ are unknown. The objective is to extract the function $x_i(\mathbf{p}, \mathbf{q})$ by making measurements of x_i at a set of values of $\mathbf{p}$ and $\mathbf{q}$, and processing them strategically to determine the function $x_i(\mathbf{p}, \mathbf{q})$. For notational simplicity, drop the suffix i which is fixed and refer to $x_i(\mathbf{p}, \mathbf{q})$ as $x(\mathbf{p}, \mathbf{q})$. We consider two situations:

(1) Determine $x(\mathbf{q})$ as a function of the sources $\mathbf{q}$, when $\mathbf{p}$ is fixed at $\mathbf{p} = \mathbf{p}_0$,
(2) Determine $x(\mathbf{p})$ as a function of $\mathbf{p}$ at a fixed value of $\mathbf{q} = \mathbf{q}_0$.

1.4.1 A Generalized Superposition Theorem

Here, we suppose that the parameter vector $\mathbf{p}$ is fixed at $\mathbf{p} = \mathbf{p}_0$, and the function $x(\mathbf{q})$ is to be determined from measurements made on the system. To proceed we make the assumption that

$$\mathbf{q} = \mathbf{v}_1 + \mathbf{v}_2 + \cdots + \mathbf{v}_k, \tag{1.53}$$

and $\mathbf{b}(\mathbf{q})$ can be decomposed as

$$\mathbf{b}(\mathbf{q}) = \mathbf{b}_1(\mathbf{v}_1) + \mathbf{b}_2(\mathbf{v}_2) + \cdots + \mathbf{b}_k(\mathbf{v}_k), \tag{1.54}$$

with

$$\mathbf{b}_j(\mathbf{v}_j)|_{\mathbf{v}_j=0} = 0, \quad j = 1, 2, \ldots, k. \tag{1.55}$$

Example 1.6. Suppose $\mathbf{b}(\mathbf{q})$ is given as

$$\mathbf{b}(\mathbf{q}) = \begin{pmatrix} q_1 \\ q_1 q_2 \\ q_3^2 \end{pmatrix}. \tag{1.56}$$

One can write $\mathbf{q}$ as

$$\mathbf{q} = \underbrace{\begin{pmatrix} q_1 \\ q_2 \\ 0 \end{pmatrix}}_{\mathbf{v_1}} + \underbrace{\begin{pmatrix} 0 \\ 0 \\ q_3 \end{pmatrix}}_{\mathbf{v_2}}, \tag{1.57}$$

giving the following decomposition for $\mathbf{b}(\mathbf{q})$

$$\mathbf{b}(\mathbf{q}) = \underbrace{\begin{pmatrix} q_1 \\ q_1 q_2 \\ 0 \end{pmatrix}}_{\mathbf{b}_1(\mathbf{v}_1)} + \underbrace{\begin{pmatrix} 0 \\ 0 \\ q_3^2 \end{pmatrix}}_{\mathbf{b}_2(\mathbf{v}_2)}, \tag{1.58}$$

which satisfies (1.55).

The assumptions (1.53), (1.54) and (1.55) allow us to state a general superposition theorem as described below. Let x^j denote the measured value of $x(\mathbf{v}_j)$, $j = 1, 2, \ldots, k$.

Theorem 1.2. (Generalized Superposition Theorem). *Under assumptions* (1.53), (1.54) *and* (1.55)

$$x(\mathbf{q}) = x^1 + x^2 + \cdots + x^k. \tag{1.59}$$

Proof. Let $\mathbf{A}^i(\mathbf{b}(\mathbf{q}))$, as before, denote the matrix $\mathbf{A}$ with the ith column replaced by $\mathbf{b}(\mathbf{q})$. Then

$$x(\mathbf{q}) = \frac{|\mathbf{A}^i(\mathbf{b}(\mathbf{q}))|}{|\mathbf{A}|}. \tag{1.60}$$

Using (1.53), (1.54) and (1.55), and the fact that

$$|\mathbf{A}^i(\mathbf{b}(\mathbf{q}))| = |\mathbf{A}^i(\mathbf{b}_1(\mathbf{v}_1))| + |\mathbf{A}^i(\mathbf{b}_2(\mathbf{v}_2))| + \cdots + |\mathbf{A}^i(\mathbf{b}_k(\mathbf{v}_k))|, \tag{1.61}$$

one can write

$$\begin{aligned} x(\mathbf{q}) =& \frac{|\mathbf{A}^i(\mathbf{b}_1(\mathbf{v}_1))| + |\mathbf{A}^i(\mathbf{b}_2(\mathbf{v}_2))| + \cdots + |\mathbf{A}^i(\mathbf{b}_k(\mathbf{v}_k))|}{|\mathbf{A}|} \\ =& x^1 + x^2 + \cdots + x^k. \end{aligned} \tag{1.62}$$

□

Further simplification results in the special case, in which

$$\mathbf{b}(\mathbf{q}) = \mathbf{b}_1 q_1 + \mathbf{b}_2 q_2 + \cdots + \mathbf{b}_m q_m. \tag{1.63}$$

In fact the following theorem is the usual version of the Superposition Theorem one sees in books on circuit theory.

Theorem 1.3. (Superposition Theorem). *Let x^j denote the solution for $x(\mathbf{q})$ when the input is*

$$\mathbf{q}_j := \begin{pmatrix} 0 \\ 0 \\ \vdots \\ q_j \\ \vdots \\ 0 \end{pmatrix}, \quad j = 1, 2, \ldots, m, \tag{1.64}$$

and $x(\mathbf{q})$ the solution for the input

$$\mathbf{q} = \begin{pmatrix} q_1 \\ q_2 \\ \vdots \\ q_m \end{pmatrix} = \mathbf{q}_1 + \mathbf{q}_2 + \cdots + \mathbf{q}_m, \tag{1.65}$$

then

$$x^j = \underbrace{\frac{|\mathbf{A}^i(\mathbf{b}_j)|}{|\mathbf{A}|}}_{\alpha_j} q_j, \tag{1.66}$$

and

$$\begin{aligned} x(\mathbf{q}) &= x^1 + x^2 + \cdots + x^m \\ &= \alpha_1 q_1 + \alpha_2 q_2 + \cdots + \alpha_m q_m. \end{aligned} \tag{1.67}$$

Furthermore, if

$$\mathbf{q} = \begin{pmatrix} c_1 q_1 \\ c_2 q_2 \\ \vdots \\ c_m q_m \end{pmatrix} = c_1 \mathbf{q}_1 + c_2 \mathbf{q}_2 + \cdots + c_m \mathbf{q}_m, \tag{1.68}$$

then

$$x(\mathbf{q}) = c_1 x^1 + c_2 x^2 + \cdots + c_m x^m. \tag{1.69}$$

Proof. The proof follows from the fact that in the case when (1.68) holds, then

$$|\mathbf{A}^i(\mathbf{b}(\mathbf{q}))| = c_1 |\mathbf{A}^i(\mathbf{b}_1)| q_1 + c_2 |\mathbf{A}^i(\mathbf{b}_2)| q_2 + \cdots + c_m |\mathbf{A}^i(\mathbf{b}_m)| q_m, \tag{1.70}$$

and so

$$x(\mathbf{q}) = c_1 \frac{|\mathbf{A}^i(\mathbf{b}_1)|}{|\mathbf{A}|} q_1 + c_2 \frac{|\mathbf{A}^i(\mathbf{b}_2)|}{|\mathbf{A}|} q_2 + \cdots + c_m \frac{|\mathbf{A}^i(\mathbf{b}_m)|}{|\mathbf{A}|} q_m. \tag{1.71}$$

$\square$

The above theorem states that the solution for the case where the sources are simultaneously active, can be recovered as the sum of the solutions corresponding to the sources acting one at a time.

Remark 1.2. In this case the q_j occur in the matrix $\mathbf{A}^i(\mathbf{b}(\mathbf{q}))$ with rank 1.

Remark 1.3. Note that in Theorem 1.2, if

$$\mathbf{q} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_k\mathbf{v}_k, \tag{1.72}$$

it is in general not true that

$$x(\mathbf{q}) = c_1x^1 + c_2x^2 + \cdots + c_kx^k. \tag{1.73}$$

This is because $\mathbf{b}(c_j\mathbf{v}_j) \neq c_j\mathbf{b}(\mathbf{v}_j)$. It is in this sense that the above two theorems are different.

Example 1.7. Recall the circuit example 1.1 where

$$\mathbf{q} = \begin{pmatrix} I \\ V \end{pmatrix} = \underbrace{I}_{q_1}\begin{pmatrix} 1 \\ 0 \end{pmatrix} + \underbrace{V}_{q_2}\begin{pmatrix} 0 \\ 1 \end{pmatrix}. \tag{1.74}$$

The current I_1 can be written as

$$I_1 = \underbrace{\frac{I_1^{1*}}{I^*}}_{\alpha_1} I + \underbrace{\frac{I_1^{2*}}{V^*}}_{\alpha_2} V, \tag{1.75}$$

where α_1 and α_2 can be determined by applying the inputs $[I^*, 0]^T$ and $[0, V^*]^T$, measuring I_1^{1*} and I_1^{2*}, and setting

$$\alpha_1 = \frac{I_1^{1*}}{I^*}, \quad \alpha_2 = \frac{I_1^{2*}}{V^*}. \tag{1.76}$$

1.4.2 A Measurement Theorem

In this section, we consider the determination of the function $x(\mathbf{p}, \mathbf{q}_0) =: x(\mathbf{p})$ when the source vector is fixed at $\mathbf{q} = \mathbf{q}_0$. We exploit the fact that in this case $x(\mathbf{p})$ has the form

$$x(\mathbf{p}) = \frac{\beta(\mathbf{p}, \mathbf{q}_0)}{\alpha(\mathbf{p})} =: \frac{\beta(\mathbf{p})}{\alpha(\mathbf{p})}, \tag{1.77}$$

where $\alpha(\mathbf{p})$ and $\beta(\mathbf{p}, \mathbf{q}_0)$ are defined as in (1.18) and (1.19):

$$\alpha(\mathbf{p}) = |\mathbf{A}(\mathbf{p})| = \sum_{i_l=0}^{r_l} \cdots \sum_{i_2=0}^{r_2} \sum_{i_1=0}^{r_1} \alpha_{i_1 i_2 \cdots i_l} p_1^{i_1} p_2^{i_2} \cdots p_l^{i_l}, \tag{1.78}$$

$$\beta(\mathbf{p}, \mathbf{q}_0) = |\mathbf{B}(\mathbf{p}, \mathbf{q}_0)| = \sum_{i_l=0}^{t_l} \cdots \sum_{i_2=0}^{t_2} \sum_{i_1=0}^{t_1} \beta_{i_1 i_2 \cdots i_l} p_1^{i_1} p_2^{i_2} \cdots p_l^{i_l}. \tag{1.79}$$

We, however, assume that for an unknown model ($\mathbf{A}(\mathbf{p})$ and $\mathbf{b}(\mathbf{q})$ not known), the ranks r_i and t_i are known. The coefficients in (1.78) and (1.79), denoted respectively by the vectors α and β, are unknown, and the number of unknown coefficients is $\mu := \prod_{i=1}^{l}(r_i + 1) + \prod_{i=1}^{l}(t_i + 1) - 1$. They can, however, be determined by setting the parameter vector $\mathbf{p}$ to μ different sets of values and solving a set of μ linear equations in the μ unknowns.

Theorem 1.4. (A Measurement Theorem). *The function $x(\mathbf{p})$ can be determined from μ measurements and solution of a system of μ linear equations in the unknown coefficient vectors α and β, called the measurement equations.*

Example 1.8. Suppose that

$$\mathbf{p} = \begin{bmatrix} p_1 \\ p_2 \end{bmatrix}, \quad \mathbf{q} = \begin{bmatrix} q_1 \\ q_2 \end{bmatrix} \quad \text{and} \quad r_1, t_1 = 1, \quad r_2, t_2 = 2.$$

Then,

$$x_i(\mathbf{p}, \mathbf{q}) = \frac{q_1 \beta_{i1}(\mathbf{p}) + q_2 \beta_{i2}(\mathbf{p})}{\alpha(\mathbf{p})}, \tag{1.80}$$

where

$$\alpha(\mathbf{p}) = \alpha_{00} + \alpha_{10} p_1 + \alpha_{01} p_2 + \alpha_{11} p_1 p_2 + \alpha_{12} p_1 p_2^2, \tag{1.81}$$

$$\beta_{ij}(\mathbf{p}) = \beta_{ij00} + \beta_{ij10} p_1 + \beta_{ij01} p_2 + \beta_{ij11} p_1 p_2 + \beta_{ij12} p_1 p_2^2, \tag{1.82}$$

for $i = 1, 2$, $j = 1, 2$. Now, if p_1, p_2, q_1, q_2 are the design parameters, then the total number of unknown coefficients in (1.80) and thus the number of measurements required to determine the parameterized solution in 5D space of $(p_1, p_2, q_1, q_2, x_i)$ will be 14. In the case where only p_1 and p_2 are the design parameters (q_1 and q_2 are fixed), the number of unknown coefficients is 9 and thus 9 measurements determine the parameterized function. In the situation where p_1 and p_2 are fixed and the design parameters are q_1 and q_2, the number of unknown coefficients becomes 2 and can be determined from two measurements. Note that in (1.80) the dependence of the solution on $\mathbf{p}$ and $\mathbf{q}$ is nonlinear, however, the function can be determined by solving a set of linear equations.

Remark 1.4. This theorem is a generalization of Thevenin's Theorem of circuit theory. This connection will be made in Chaps. 2 and 3. The result given here however applies to any linear system, be it electrical, mechanical, hydraulic and so on.

1.5 Notes and References

In this chapter we developed some formulas expressing the solutions of parameterized sets of linear equations, showing explicitly the nature of the parameter dependence. We also developed a proof of the Superposition Theorem for general linear systems and presented a theorem on the number of measurements required to extract a parameterized solution function for an unknown linear system. These formulas will be useful in subsequent chapters to develop a standardized measurement based approach to the design of linear systems, in various branches of engineering. Some of these applications have been recently explored in [1–6].

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