HANDBOOK *of* THE HISTORY OF LOGIC

VOLUME 11

LOGIC: A HISTORY OF ITS CENTRAL CONCEPTS

Edited by

Dov M. Gabbay, Francis Jeffry Pelletier, and John Woods

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Handbook of the History of Logic

General Editors

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CONTENTS

PREFACE

The present volume marks the conclusion of the *Handbook of the History of Logic* series. This capstone volume addresses central topics in the history of logic, showing how logicians, philosophers, mathematicians and others understood these topics over the years and how they guided their development down to the present century.

Certainly the most central topic in logic is the notion of *logical consequence.* Asmus and Restall start with Aristotle's definition of a syllogism as "an argument in which, certain things having been assumed, something other than these follows of necessity from their truth, without needing any term from outside" and carry the explanation of this conception through the middle ages and into the twenty-first century. Any account of logical consequence must determine the type of entities that can be premises and conclusion, must explain what ways premises can combine, and crucially, must explain the types of connection that that are allowed to hold between premises and conclusion in order for it to really be a consequence.

A part of this explanation will involve certain connected concepts: the quantifiers and the connectives. Bonevac traces the notion of quantification from Aristotle through modern generalized quantifiers. It is important to note, he says, that there is no theory-neutral way of defining quantification or even of delineating the class of quantifiers, and so a history of quantification has to trace the development of both *what* is to be explained along with *how* it is to be explained. Alongside the account of quantifiers and quantification needs to be an account of the logical particles — the connectives. Bonevac and Dever discuss the implicit treatment of propositional connectives in Aristotle before moving on to the explicit theory of them developed by the Stoics. The development of an understanding of the connectives took a winding path from the Stoics through the medieval *logica vetus* (Old Logic) and the revolutionary *logica nova* (New Logic) of the 13$^{\text{th}}$ century, through the under-appreciated algebraic understanding of Leibniz, and to the "Modern-Era logicians" of the 19$^{\text{th}}$ and 20$^{\text{th}}$ century.

As Bonevac and Dever remark, the history of the connectives is marked by an ambivalence between the attitude that the connectives are operators on the *content* of the items being connected and an attitude that they are operators on the *speech-act force* of the items (say, in a *presentation* of an argument). And again, there is the ambivalence between the view that negation is a *propositional* operator ("It is not the case that –") and that it is a *term* operator ("– is not-pale"). The 20$^{\text{th}}$ century saw the latter issue decided in favor of the propositional approach for negation. However, the question of whether negation is a content or a speech-act operator (for instance, denial) is disputed. Speranza and Horn start with Paul Grice's account of negation, using it as a springboard to discuss the ways this difference has been in effect over the history of logic.

Aristotle's notion of "following of necessity from the truth of the premises" is often described in terms of the truth-values of the premises and the conclusion: It is not possible for the truth-value of every premise to be true while the truth-value of the conclusion is false. But of course, the history of logic has seen accounts where there are more than two truth-values, and furthermore where there are "gaps" and "gluts" of truth-value. Indeed, the notion of truth-value permeates much of the broader realms of philosophy, linguistics, mathematics and computer science, and Béziau undertakes a very broad-ranging discussion of the "mathematical conception of truth-value" to show how it underlies many of the more familiar conceptions that are associated with that concept.

Modality is yet another central concept in logic. Not only is it employed in Aristotle's definition of a correct argument, but also it features in the characterization of modalized sentences, and thereby into metaphysics and language (*de re* and *de dicto* modalities). Knuuttila explores the ancient and medieval traditions in modality — distinguishing modality as "extensional" (all possibilities will be actualized) from modality as "alternativity" — and showing where these two conceptions emerge in more recent accounts of modality. These differing accounts also are manifested in the modal syllogistic and logics that were developed in ancient and medieval times, Knuuttila shows, as were interpretations of the modalities in terms of epistemic operators like *knows* and *believes*.

One version of logic employs no independently-claimed-to-be-necessarily-true statements (axioms) but instead employs only rules. Although some have claimed that Aristotle's syllogistic is such a system, the more modern version traces its history to 1934. Despite this very recent invention, most logic that is currently taught in philosophy is of this nature — "natural deduction". Pelletier and Hazen discuss the history (since 1934) of this development, its relationship with other conceptions of logic, and the metatheoretic facts that allow it to have such a prominent position in modern logic.

A rather different conception of logic arises when one thinks of sentential implication as the basic operation, and thinks of that operation as asserting some sort of "connection" between the antecedent and consequent. McCall traces this conception from the Stoics (and also Aristotle) through its development in the middle ages, to Ramsey, Nelson and Angell in the 20$^{\text{th}}$ century and the interpretation of connexive implication within relevant logics. McCall also displays the results of some empirical studies that seem to favor a "connexive interpretation" of *if–then*.

The notion of logical type was brought into logical prominence with the publication of *Principia Mathematica* in 1910, although as Kamareddine, Laan, and Nederpelt show, the notion was always present in mathematics before then. They trace the development of the theory of types from Russell-Whitehead to Church's simply-typed λ-calculus of 1940, and they show how the logical paradoxes that entered into the formal systems of Frege, Cantor and Peano brought forth the first explicit theory of types in Russell's *Principles of Mathematics*.

A wider notion of logic involves the contrast between good arguments and merely good-seeming arguments. A good argument need not be deductively valid, as we all learn in elementary practical logic; and furthermore, some deductively valid arguments are not good arguments in this sense (e.g., the ones that are obviously circular). So another part of logic that has been passed down to us over the ages is the study of the distinction between good and merely good-seeming arguments. Aristotle was the first to codify this (in the *Topics* and *Sophistical Refutations*), just as he was the first to codify the notion of deductive correctness (in the *Prior Analytics*). Woods traces the evolving notion of a fallacy in argumentation from its Aristotelian beginnings through the late 20$^{\text{th}}$ century.

The final topic in this survey of the central topics in the history of logic is the use of diagrams in logical reasoning. Moktefi and Shin survey the very well-known Euler diagrams, showing their modification by Venn and Peirce. Additionally, other diagrammatic traditions — e.g., the use of tables and linear diagrams — are surveyed. With regard to the full predicate logic, Moktefi and Shin explore Frege's two-dimensional graphical notation and Peirce's Existential Graphs. They also address the important question of the place of diagrams as representational systems in their own right, and the possibility of having rules of inference that directly characterize logical consequence in such a visual representation scheme.

The Editors are in the debt of the volume's superb authors. For support and encouragement thanks are also due Paul Bartha, Head of Philosophy at UBC, and Christopher Nicol, Dean of Arts and Science, and Kent Peacock, Chair of Philosophy, both at the University of Lethbridge.

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A HISTORY OF THE CONSEQUENCE RELATIONS

Conrad Asmus and Greg Restall

1 INTRODUCTION

Consequence is a, if not *the*, core subject matter of logic. Aristotle's study of the syllogism instigated the task of categorising arguments into the logically good and the logically bad; the task remains an essential element of the study of logic. In a logically good argument, the conclusion follows validly from the premises; thus, the study of consequence and the study of validity are the same.

In what follows, we will engage with a variety of approaches to consequence. The following neutral framework will enhance the discussion of this wide range of approaches. Consequences are conclusions of valid arguments. Arguments have two parts: a conclusion and a collection of premises. The conclusion and the premises are all entities of the same sort. We will call the conclusion and premises of an argument the *argument's components* and will refer to anything that can be an argument component as a *proposition*. The class of propositions is defined functionally (they are the entities which play the functional role of argument components); thus, the label should be interpreted as metaphysically neutral. Given the platonistic baggage often associated with the label "proposition", this may seem a strange choice but the label is already used for the argument components of many of the approaches below (discussions of Aristotlean and Medieval logic are two examples). A consequence relation is a relation between collections of premises and conclusions; a collection of premises is related to a conclusion if and only if the latter is a consequence of the former.

Aristotle's and the Stoics' classes of arguments were different, in part, because their classes of propositions differed. They thought that arguments were structures with a single conclusion and two or more premises;[1] conclusions and premises (that is, propositions) were the category of things that could be true or false. In Aristotlean propositions, a predicate is applied to a subject; the Stoics allowed for the *recombination* of propositions with connectives. Later on, some medieval logicians restricted propositions to particular concrete tokens (in the mind, or spoken, or written).

[1]This is until Antipater, head of the Stoic school around 159 – 130 BCE, who "recognized inference from one premise, his usage was regarded as an innovation" [Kneale and Kneale, 1962, p 163].

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Changing the class of realisers of the propositional functional role affects the consequence relation. A relation involving only abstract propositions must differ from a relation which involves some of concrete realisers. Not every change in the composition of propositions, however, is equal. If there is a mapping that connects the abstract propositions with the concrete sentences, and the consequence relation on these collections respects this mapping, then the differences are more metaphysical than they are logical. If there is no such mapping, then the choice between these implementations is of serious logical importance.

Aristotle and the Stoics dealt with arguments with two or more premises. Without further investigation of historical details, this can be interpreted in two ways: (1) any argument with fewer than two premises is invalid, or (2) arguments cannot have fewer than two premises. On the first interpretation, there is some necessary requirement for validity that zero and one premise arguments always fail to satisfy. According to some schools: for a conclusion to be a consequence of the premises, it must be genuinely *new*. This makes all single premise arguments invalid. Similarly, a zero premise argument is not one where the conclusion *results from* the premises. This is a choice about what the consequence relation is: whether a consequence has to be new, whether it must result from the premises, and so on. Different approaches to this issue have be taken through the history of logical consequence. Sometimes a rigid adherence to the motivations of a consequence being *new* and *resulting from* premises is maintained; at other times, this is sacrificed for the sake of simplicity and uniformity.

The second interpretation limits how a collection of premises can be structured in an argument. The combination of two propositions (one as a premise and the other as a conclusion) isn't a good argument because it isn't an argument. Premise combination has often been treated rather naively. Recently, careful discussions of premise combination have come out of Gentzen's proof systems and substructural logic. In substructural logics, premises are not combined as unordered sets. Different structural restrictions on the combination of premises, and the ways one is able to manipulate them (structural rules), result in different consequence relations. There has also been a loosening in the forms that conclusions take. Typical arguments *seem* to have exactly one conclusion (see [Restall, 2005] for an argument against this). This led to a focus on single conclusions as consequences of premises. More generally, however, we can investigate whether a *collection* of conclusions is a consequence of a collection of premises.

Any theorist of consequence needs to answer the following questions:

1. What sort of entity can play the role of a premise or of a conclusion? That is, what are propositions?

2. In what ways can premises combine in an argument? In what ways can conclusions combine in an argument?

3. What connection must hold between the premises and the conclusion(s) for the conclusion(s) be a consequence of the premises?

An answer to the first question has two main parts. There is the form of propositions (for example, on Aristotle's view propositions always predicate something of a subject) and the composition of propositions (for example, on a medieval nominalist's theory of propositions they are concrete singulars).

There are two broad approaches to the third question. Some theorists focus on a property of propositions; some theorists focus on connections between conclusions and premises. In both cases, consequence is explicated in terms of something else. In the first approach, the conclusion is a consequence of the premises if and only if, whenever the premises have some specified property, so does the conclusion. This approach focusses on whether the premises and conclusion have the designated property or not, it doesn't *rely* on a strong connection between premises and conclusion. In the paradigmatic example, this property is truth. The second approach is more concerned with the relation between the premises and conclusion. The consequence relation is build on top of another relation between premises and conclusions. If the premises and conclusion of an argument are connected by any number of steps by the basic relation, then the conclusion is a consequence of the premises. Paradigmatic examples are based on proof theories. We will refer to the first type of approaches as property based approaches, and the second as transference based approaches. There are many hybrids of the two approaches. A truth preservation approach sounds like a property based approach, but this depends on what we make of *preservation*. If it is important that the truth of the conclusion is connected via a processes of transference to the truth of the premises, then the approach has both property and transference features.

Different answers to these three questions originate from a variety of sources. Sometimes answers (especially to the first question) come from metaphysics; sometimes answers (especially to the third question) come from epistemology. Importantly, different answers are connected to different properties that consequence relations are expected to have. In the next three sections, we will look at some features that have struck theorists as important properties for consequence relations. Different answers to the three questions often correspond to different emphases on these properties.

Theorists, like Tarski in the quote below, have been aware that there are many tensions in developing an account of consequence. There is usually a trade off between precision, adherence to everyday usage of the concept, and with adherence to past accounts. Any precise account will be, to some extent, revisionary. In [Tarski, 1956b, p 409] Tarski says,

> The concept of *logical consequence* is one of those whose introduction into the field of strict formal investigation was not a matter of arbitrary decision on the part of this or that investigator; in defining this concept, efforts were made to adhere to the common usage of the language of everyday life. But these efforts have been confronted with the difficulties which usually present themselves in such cases. With respect to the clarity of its content the common concept of consequence is in no way superior to other concepts of everyday language. Its extension

> is not sharply bounded and its usage fluctuates. Any attempt to bring into harmony all possible vague, sometimes contradictory, tendencies which are connected with the use of this concept, is certainly doomed to failure. We must reconcile ourselves from the start to the fact that every precise definition of this concept will show arbitrary features to a greater or less degree.

This leaves the theorist with a final question to answer: What is the point of the theory? Precision, accord with everyday usage, accord with the normative constraints on reasoning, and many other answers have been forthcoming in the history of logical consequence.

1.1 Necessity and Counterexamples

Aristotle categorised syllogisms into those that are deductions and those that are not. The distinguishing feature of a deduction is that the conclusion *necessarily* results from the premises. That consequences follow of necessity from premises was one of the earliest characteristic features of consequence to be emphasised. It is not always easy to determine, however, what substance theorists impart into this necessity. The way in which theorists categorise arguments provides insight into how they understand necessity.

Aristotle, the Stoics, the medievals, Leibniz, Kant, and many more of the logicians and philosophers dealt with in this entry discuss necessity and modal logic. Of particular importance is Leibniz's account of necessity. A proposition is necessary if it is true in all possible worlds. There are two important parts of this move. Firstly, the notion of *possible world* is introduced. Possible worlds can serve as a type of *counterexample*. If it is possible for the premises of an argument to be true, and the conclusion false, then this is taken to demonstrate that the argument is invalid, and thus that the conclusion is not a consequence of the premises. Secondly, necessity is fixed as truth in *every* possible world. Universal quantification over possible worlds is a genuine advancement: for example, consider the equivalence of $\Box(A \wedge B)$ with $\Box A \wedge \Box B$.

A conclusion that is a consequence of a collection of premises should hold in *any* situation in which the premises do. Logical consequence can be used to reason about hypothetical cases as well as the actual case; the conclusion of a good argument doesn't merely follow *given the way things are* but will follow *no matter how things are.*

A characterisation of logical consequence in terms of necessity can lead away from the transference approach to consequence. A demonstration that there are no counterexamples to an argument needn't result in a recipe for the connecting the premises and conclusion in any robust sense. Necessity is not, however, anathema to the transference approach. If the appropriate emphasis is placed in "necessarily *results from*" and "consequences *follow* of necessity", and this is appropriately implemented, then transference can still be respected.

1.2 Formality and Structure

Necessity is not sufficient for *logical* consequence. Consider the argument:

> All logicians are blue.
>
> Some blue objects are coloured.
>
> Therefore, all logicians are coloured.

It seems that, if the premises of the argument are true, the conclusion *must* also be; the conclusion seems to follow of necessity from the first premise. This is not a *formally* valid argument. That the conclusion is necessitated *relies* on all blue objects being coloured. This reliance disqualifies it as a logical consequence. A conclusion is a formal consequence of a collection of premise not when there is merely no possibility of the premises being true and conclusion false, but when it has an *argument form* where there is no possibility of *any* instance of the form having true premises and a false conclusion. Counterexamples are not only counterexamples to arguments but to argument forms.

In this example, there are counterexamples to the argument form:

> All αs are βs.
>
> Some βs are γs.
>
> Therefore, all αs are γs.

If the argument is not an instance of any other valid argument form, it is not valid and the conclusion is not a formal consequence of the premises. Argument forms and instances of argument forms play a crucial role in logical consequence; in some ways they are more central than arguments. Logical consequence is *formal* in at least this respect.

Formal consequence is not the only relation of consequence that logicians have studied. Some logicians have placed a high level of importance on *material* consequence. A conclusion is a material consequence of a collection of premises if it follows either given the way things are (so not of necessity) or follows of necessity but not simply because of the form of the argument. In order to properly distinguish material and formal consequence we require a better characterisation of the forms of propositions and of arguments.

That logical consequence is schematic, and in this sense formal, is a traditional tenet of logical theory. There is far more controversy over other ways in which consequence may be formal. The use of schemata is not sufficient for ruling out the sample argument about blue logicians. The argument appears to be of the following form:

> $(\forall x)(Lx \rightarrow x \text{ is blue})$
>
> $(\exists x)(x \text{ is blue} \wedge x \text{ is coloured})$

Therefore, $(\forall x)(Lx \to x \text{ is coloured})$,

where L is the only schematic letter. There are no instances of this schema where it is possible for the premises of the argument to be true and the conclusion false. Whether this counts as a legitimate argument form depends on what must be, and what may be, treated schematically. This choice, in turn, rests on the other ways in which consequence is formal.

Sometimes logic is taken to be "concerned merely with the form of thought" [Keynes, 1906, p 2]. This can be understood in a number of ways. Importantly, it can be understood as focussing on the general structure of propositions. If propositions have some general form (a predicate applied to a subject, has some recursive propositional structure, and so on) then consequence is formal in that it results from the logical connections between these forms. In MacFarlane's discussion of the formality of logic, this is described as (1) "logic provides constitutive norms for thought as such" [MacFarlane, 2000, p ii]. The other two ways in which logic can be formal what MacFarlane points out are:

(2) logic is "indifferent to the particular identities of objects."

(3) logic "abstracts entirely from the *semantic content* of thought."

He argues, convincingly, that Kant's logic was formal in all three senses, but that later theorists found themselves pressured into choosing between them.

1.3 A Priori *and Giving Reasons*

Logical consequence is often connected to the practice of reason giving. The premises of a valid argument are reasons for the conclusion. Some transference approaches take logical consequence to rest on the giving of reasons: C is a consequence of the premises Δ if and only if a justification for C can be constructed out of justifications for the premises in Δ. Logical consequence, on this view, is about the transformation of reasons for premises into reasons for conclusions.

Most reason giving doesn't rely entirely on logical consequence. Lots of reasoning is *ampliative*; the conclusion genuinely says more than the combination of the premises. The common example is that *there is smoke* is a reason for that *there is fire.* The argument:

There is smoke.

Therefore, there is fire.

is invalid — the conclusion is not a logical consequence of the premise. It is a *material consequence* of the premise. In this entry, we ill focus on *logical* consequence. In logical reason giving, the reasons are *a priori* reasons for the conclusion. That the premises of a valid argument are reasons for the conclusion does not rely on any further evidence (in this example, regarding the connections between smoke and fire).

Some rationalists, the rationalistic pragmatists, hold that material consequence is also, in some sense, *a priori* (e.g. Sellars [Sellars, 2007, especially p 26]). Material consequences are, however, not *necessary* in the same way. A counterexample to a material consequence does not immediately force a revision of our conceptual scheme on us. This is not true with logical consequence: either the purported counterexample must be rejected, or the purported logical consequence must be. This necessity is closely connected to the normativity of logical and material consequence. I can believe that there is smoke and that there isn't fire, *so long as* I also believe that this is an exceptional situation. There is no similar exception clause when I accept the premises of an instance of *modus ponens* and reject its conclusion.

The connection between logical consequence and the giving of reasons highlights the normative nature of consequence. If an argument is valid and I am permitted to accept the premises, then I am permitted to accept the conclusion. Some theoriests make the stronger claim that if one accepts the premises of a valid argument, then one ought to accept the conclusion. One of the many positions between these positions is that if one accepts the premises of a valid argument, then one ought not reject the conclusion.

A focus on the giving of reasons and the normativity of logical consequence is often the result of an aim to connect logical consequence to human activity — to concrete cases of reasoning. Logical consequence, from this perspective, is the study of a particular way in which we are obligated and entitled to believe, accept, reject and deny.

2 ARISTOTLE [384 BCE–322 BCE]

Aristotle's works on logic are the proper place to begin any history of consequence. They are the earliest formal logic that we have and have been immensely influential. Kant is merely one example of someone who thought that Aristotle's logic required no improvement.

> It is remarkable also that to the present day this logic has not been able to advance a single step, and is thus to all appearance a closed and completed body of doctrine. If some of the moderns have thought to enlarge it . . . , this could only arise from their ignorance of the peculiar nature of logical science.
>
> [Kant, 1929, Bviii–ix]

Aristotle categorised syllogisms based on whether they were *deductions*, where the conclusion is a consequence of the premises.

According to Aristotle, propositions are either simple — predicating a property of a subject in some manner — or can be analysed into a collection of simple propositions. There are three parts to any simple proposition: subject, predicate and kind. In non-modal propositions predicates are either affirmed or denied of the subject, and are affirmed or denied either in part or universally (almost everything is controversial in the modal cases).

Subjects and predicates are *terms*. Terms come in two kinds: universal and individual. Universal terms can be predicates and subjects (for example: children, parent, cat, weekend). Individual terms can only be the subject of a proposition (for example: Plato, Socrates, Aristotle). A proposition which seems to have a individual term in the predicate position is, according to Aristotle, not a genuine proposition but merely an accidental predication that depends on a genuine predication for its truth (for example, "The cat on the mat is Tully" depends on "Tully is on the mat").

A proposition can be specified by nominating a subject, a predicate and a kind. Here are some examples with universal terms and the four non-modal kinds: universal affirmation, partial affirmation, universal denial and partial denial:

Example	Kind	Code
All *children* are *happy*.	Universal Affirmative	A
No *weekends* are *relaxing*.	Universal Negative	E
Some *parents* are *tired*.	Particular Affirmative	I
Some *cats* are not *friendly*.	Particular Negative	O

Any collection of propositions is a syllogism; one proposition is the conclusion and the rest are premises. Aristotle gives a well worked out categorisation of a subclass of syllogisms: the *categorical* syllogisms. A categorical syllogism has exactly two premises. The two premises share a term (the middle term); the conclusion contains the other two terms from the premises (the extremes). There are three resulting figures of syllogism, depending on where each term appears in each premise and conclusion. Each premise and conclusion (in the non-modal syllogisms) can be one of the four kinds in the table above.

The syllogisms are categorised by whether or not they are deductions.

> A deduction is a discourse in which, certain things having been supposed, something different from the things supposed results of necessity because these things are so. By 'because these things are so', I mean 'resulting through them,' and by 'resulting through them' I mean 'needing no further term from outside in order for the necessity to come about.' [Smith, 1989, Prior Analytics A1:24b]

The following example is a valid syllogism in the second figure with E and I premises and an O conclusion (it has come to be called "Fest$i$no").

> No weekends are relaxing.
>
> Some holidays are relaxing.
>
> Therefore, some holidays are not weekends.

Aristotle categorises syllogisms based on their form. He justifies this particular argument's form:

> No Bs are As.
>
> Some Cs are As.
>
> Therefore, some Cs are not Bs.

in a two step procedure. Aristotle transforms the argument form by converting the premise "No Bs are As" into the premise "No As are Bs". This transforms the second figure Festino into the first figure Ferio. The justification of Festino rests on the justification of the conversion and the justification of Ferio. Here is Aristotle's justification of the former:

> Now, if A belongs to none of the Bs, then neither will B belong to any of the As. For if it does belong to some (for instance to C), it will not be true that A belongs to none of the Bs, since C is one of the Bs. [Smith, 1989, Prior Analytics A2:25a]

There is no justification for the latter: merely an assertion that the conclusion follows of necessity.

Aristotle uses a counterexample to show that the syllogistic form:

> All Bs are As
>
> No Cs are Bs
>
> Therefore, All Cs are As

is invalid. He reasons in the following way:

> However, if the first extreme [A] follows all the middle [B] and the middle [B] belongs to none of the last [C], there will not be a deduction of the extremes, for nothing necessary results in virtue of these things being so. For it is possible for [A] to belong to all as well as to none of the last [C]. Consequently, neither a particular nor a universal conclusion becomes necessary; and, since nothing is necessary because of these, there will not be a deduction. Terms for belonging to every are animal, man, horse; for belonging to none, animal, man, stone. [Smith, 1989, Prior Analytics A4:26a]

Aristotle concludes that the argument form is not a deduction as the syllogism:

> All men are animals
>
> No stones are men
>
> Therefore, All stones are animals

is of the same form but one has true premises and a false conclusion, so the conclusion of the other syllogism cannot follow of necessity.

3 STOICS [300 BCE–200 CE]

The Stoic school of logicians provided an alternative to Aristotle's logic. The Stoic school grew out of the Megarian and Dialectical schools.[2] The Megarians and the members of the Dialectical school contributed to the development of logic by their attention to paradoxes, a careful examination of modal logic and by debating the nature of the conditional (notably by Philo of Megara). Eubulides was particularly noted among the Megarians for inventing paradoxes, including the liar paradox, the hooded man (or the Electra), the sorites paradox and the horned man. As we will return to the lair paradox when discussing the medieval logicians, we will formulate it here. "A man says that he is lying. Is what he says true or false?" [Kneale and Kneale, 1962, p 114]. If the man says something true, then it seems that he is indeed lying — but if he is lying he is not saying something true. Similarly, if what the man says is false, then what he says is not true and, thus, he must be lying — but he says that he is lying and we have determined that he is lying, so what he says is true. Diodorus Cronus is well know for his *master argument*. Diodorus' argument is, plausibly, an attempt to establish his definition of modal notions.

According to Epictetus:

> The Master Argument seems to have been formulated with some such starting points as these. There is an incompatibility between the three following propositions, "Everything that is past and true is necessary", "The impossible docs not follow from the possible", and "What neither is nor will be is possible". Seeing this incompatibility, Diodorus used the convincingness of the first two propositions to establish the thesis that nothing is possible which neither is nor will be true. [Kneale and Kneale, 1962, p 119]

The reasoning involved in the argument is clearly non-syllogistic and the modal notions involved are complex.

The Stoic school was founded by Zeno of Citium, succeeded in turn by Cleanthes and Chrysippus. The third of these was particularly important for the development on Stoic logic. Chrysippus produced a great many works on logic; we encourage the reader to look at the list of works that Diogenes Laertius attributes to him [Hicks, 1925, pp 299 – 319].

A crucial difference between the Stoic and Aristotelean schools is the sorts of propositional forms they allowed. In Aristotle's propositions, a predicate is affirmed or denied of a subject. The Stoics allowed for complex propositions with a recursive structure. A proposition could be basic or could contain other propositions put together with propositional connectives, like the familiar *negation*,

[2]There is some controversy regarding who belongs to which school. The members of the Dialectical group were traditionally thought of as Megarians. For our discussion, the most noticeable of these are Philo of Megara and Diodorus Cronus. In our limited discussion it will not hurt to consider the groups as one.

conditional, *conjunction* and *disjunction*, but also the connectives *Not both ... and ...*; *... because ...*; *... rather than ...* and others. The Stoics had accounts of the meaning and truth conditions of complex propositions. This came close to modern truth table accounts of validity but, while meaning and truth were sometimes dealt with in a truth-table-like manner, validity was not.

Chrysippus recognised the following five indemonstrable moods of inference, [Kneale and Kneale, 1962, p 163] [Bury, 1933, *Outlines of Pyrrhonism* II. 157f]:

1. If the first, then the second; but the first; therefore the second.
2. If the first, then the second; but not the second; therefore not the first.
3. Not both the first and the second; but the first; therefore not the second.
4. Either the first or the second; but the first; therefore not the second.
5. Either the first or the second; but not the second; therefore the first.

These indemonstrable moods could be used to justify further arguments. The arguments, like Aristotle's categorical syllogisms, have two premises. The first premise is always complex. Notice that, even though the Stoics had a wide range of propositional connectives, only the conditional, disjunction and negation conjunction and (possibly) negation appear in these indemonstrables. This is an example of a transference style approach to logical consequence.

4 MEDIEVALS [476 CE–1453 CE]

Logic was a foundational discipline during the medieval period. It was considered to have intrinsic value and was also regarded as an important groundwork for other academic study. Medieval logic is often divided into two parts: the old and the new logic. The demarcation is based on which Aristotelian texts were available. The old logic is primarily based on Aristotle's *Categories* and *De interpretatione* (this includes discussions on propositions and the square of opposition, but importantly lacks the *prior analytics*, which deals with the syllogism) while the new logic had the benefit of the rest of Aristotle's *Organon* (in the second half of the 12^{th} century). Many medieval logicians refined Aristotle's theory of the syllogism, with particular attention to his theory of modal logic. The medieval period, however, was not confined to reworking ancient theories. In particular, the *terminist* tradition produced novel and interesting directions of research. In the later medieval period, great logicians such as Abelard, Walter Burley, William of Ockham, the Pseudo-Scotus, John Buridan, John Bradwardine and Albert of Saxony made significant conceptual advances to a range of logical subjects.

It is not always clear *what* the medieval logicians were doing, nor *why* they were doing it [Spade, 2000]. Nevertheless, it is clear that consequence held an important place in the medieval view of logic, both as a topic of invesitagation and as a tool to use in other areas. Some current accounts of logical consequence have remarkable similarities to positions from the medieval era. It is particularly interesting that early versions of standard accounts of logical consequence were considered *and rejected* by thinkers of this period (in particular, see Pseudo-Scotus and Buridan below).

The medievals carried out extensive logical investigations in a broad range of areas (including: inference and consequence, grammar, semantics, and a number of disciples the purpose of which we are still unsure). This section will only touch on three of these topics. We will discuss theories of consequentiæ, the medieval theories of consequence. We will describe how some medievals made use of consequence in solutions to insolubilia. Lastly, we'll discuss the role of consequence in the medieval area of obligationes. This third topic is particularly obscure; it will serve as an example of where consequence plays an important role but is not the focus of attention.

4.1 Consequentiæ

The category of consequentiæ was of fluctuating type. It is clear that in Abelard's work a consequentiæ was a true conditional but that in later thinkers there was equivocation between true conditionals, valid one premise arguments, and valid multiple premise arguments. This caused difficulties at times but what is said about consequentiæ is clearly part of the history of logical consequence.

The medievals broadened the range of inferences dealt with by accounts of consequentiæ from the range of consequences that Aristotle and the Stoics considered. In the following list, from [Kneale and Kneale, 1962, pp 294 – 295], items (3), (4), (9) and (10) are particularly worth noting:

1. From a conjunctive proposition to either of its parts.
2. From either part of a disjunctive proposition to the whole of which it is a part.
3. From the negation of a conjunctive proposition to the disjunction of the negations of its parts, and conversely.
4. From the negation of a disjunctive proposition to the conjunction of the negations of its parts, and conversely.
5. From a disjunctive proposition and the negation of one of its parts to the other part.
6. From a conditional proposition and its antecedent to its consequent.
7. From a conditional proposition and the negation of its consequent to the negation of its antecedent.
8. From a conditional proposition to the conditional proposition which has for antecedent the negation of the original consequent and for consequent the negation of the original antecedent.
9. From a singular proposition to the corresponding indefinite proposition.

10. From any proposition with an added determinant to the same without the added determinant.

Like Aristotle and the Stoics, the medievals investigated the logic of modalities. The connections they drew between modalities, consequentiæ and the "follows from" relation are interesting. Ockham gives us the rules [Kneale and Kneale, 1962, p 291]:

1. The false never follows from the true.
2. The true may follow from the false.
3. If a consequentiæis valid, the negative of its antecedent follows from the negative of its consequent.
4. Whatever follows from the consequent follows from the antecedent.
5. If the antecedent follows from any proposition, the consequent follows from the same.
6. Whatever is consistent with the antecedent is consistent with the consequent.
7. Whatever is inconsistent with the consequent is inconsistent with the antecedent.
8. The contingent does not follow from the necessary.
9. The impossible does not follow from the possible.
10. Anything whatsoever follows from the impossible.
11. The necessary follows from anything whatsoever.

Unlike the Stoics approach to "indemonstrables", the medievals provided analyses of consequentiæ. According to Abelard, consequentiæ form a sub-species of *inferentia*. An inferentia holds when the premises (or, in Abelard's case the antecendent) necessitate the conclusion (consequence) in virtue of their meaning (in modern parlance, an inferentia is an entailment, and the "in virtue of" condition makes the relation *relevant*). The inferentia are divided into the perfect and the imperfect. In perfect inferentia, the necessity of the connection is based on the structure of the antecendent — "if the necessity of the consecution is based on the arrangement of terms regardless of their meaning" [Boh, 1982, pp 306]. The characteristic features of perfect inferentia are remarkably close to Balzano's analysis of logical consequence.

Buridan and Pseudo-Scotus

Buridan, Pseudo-Scotus and other medieval logicians argued against accounts of consequence that were based on necessary connections. Pseudo-Scotus and Buridan provide apparent counterexamples to a range of definitions of consequence. In this section, we look at three accounts of consequence and corresponding purported counterexamples. (We rely heavily on [Boh, 1982] and [Klima, 2004].)

The first analysis we consider is: