



THE FRANK J. FABOZZI SERIES

# MATHEMATICAL METHODS FOR FINANCE

*Tools for Asset and Risk Management*

SERGIO M. FOCARDI • FRANK J. FABOZZI • TURAN G. BALI

WILEY

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# Mathematical Methods for Finance

*Tools for Asset  
and Risk Management*

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FRANK J. FABOZZI  
TURAN G. BALI

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*To the memory of my parents*  
SMF

*To my wife, Donna,  
and my children, Patricia, Karly, and Francesco*  
FJF

*To my wife, Mehtap,  
and my son, Kaan*  
TGB

# Preface

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Since the pioneering work of Harry Markowitz in the 1950s, mathematical tools drawing from the fields of standard and stochastic calculus, set theory, probability theory, stochastic processes, matrix algebra, optimization theory, and differential equations have increasingly made their way into finance. Some of these tools have been used in the development of financial theory, such as asset pricing theory and option pricing theory, as well as like theories in the practice of asset management, risk management, and financial modeling.

Different areas of finance call for different mathematics. For example, asset management, also referred to as investment management and money management, is primarily concerned with understanding hard facts about financial processes. Ultimately, the performance of an asset manager is linked to an understanding of risk and return. This implies the ability to extract information from time series data that are highly noisy and appear nearly random. Mathematical models must be simple, but with a deep economic meaning. In other areas, the complexity of instruments is the key driver behind the growing use of sophisticated mathematics in finance. There is the need to understand how relatively simple assumptions on the probabilistic behavior of basic quantities translate into the potentially very complex probabilistic behavior of financial products. Examples of such products include option-type financial derivatives (such as options, swaptions, caps, and floors), credit derivatives, bonds with embedded option-like payoffs (such as callable bonds and convertible bonds), structured notes, and mortgage-backed securities.

One might question whether all this mathematics is justified in finance. The field of finance is generally considered much less accurate and viable than the physical sciences. Sophisticated mathematical models of financial markets and market agents have been developed but their accuracy is questionable to the point that the recent global financial crisis is often blamed on unwarranted faith in faulty mathematical models. However, we believe that the mathematical handling of finance is reasonably successful and models are not to be blamed for this crisis. Finance does not study laws of nature but complex human artifacts—the financial markets—that are designed to be largely uncertain. We could make financial markets less uncertain and, thereby, mathematical models more faithful by introducing more rules and collecting more data. Collectively, we have decided not to do so and therefore, models can only be moderately accurate. Still, they offer a valuable design tool to engineer our financial systems. Nevertheless, the mathematics of finance cannot be that of physics. It is the mathematics of learning and complexity, similar to the mathematics used in studying biological and ecological systems.

In 1960, the physicist Eugene Wigner, recipient of the 1962 Nobel Prize in Physics, wrote his now famous paper “The Unreasonable Effectiveness of Mathematics in the Natural Sciences.” Wigner argued that the success of mathematics in describing natural phenomena is so extraordinary that it is in itself a phenomenon that needs explanation.<sup>1</sup> Mathematics in finance is reasonably effective and the reasons why it is reasonably effective deserve an explanation. Recently, the world went through the worst financial and economic crisis since the Great Depression. Many have pointed their fingers at the growing use of mathematics in finance and the resulting mathematical models. We would argue that mathematics does not have much to do with that crisis. In a nutshell, we believe that mathematics is reasonably effective in finance because we apply it to study large engineered artifacts—financial markets—that have been designed to have a lot of freedom. Modern financial systems are designed

be relatively unpredictable and uncontrolled in order to leave possibilities of changes and innovation. The level of unpredictability and control is different in different systems. Some systems are prone to crises. Mathematics does a reasonably good job to describe these systems. But the mathematics involved is not the same as that of physics. It is the mathematics of learning and complexity. Mathematics can be perceived as ineffective in finance only if we insist on comparing it with physics.

There are differences between finance and the physical sciences. In the three centuries following the publication of Newton's *Principia* in 1687, physics has developed into an axiomatic theory. Physical theories are axiomatic in the sense that the entire theory can be derived through mathematical deduction from a small number of fundamental laws. Physics is not yet completely unified but the different disciplines that make the body of physics are axiomatic. Even more striking is the fact that physical phenomena can be approximately represented by computational structures, so that physical reality can be mimicked by a computer program.

Though it is clear that finance has made progress and will make additional progress only by adopting the scientific method of empirical science, it should be clear that there are significant differences between finance and physics. We can identify, albeit with some level of arbitrariness, four major differences between finance and the physical sciences:

1. Finance must study a global financial system without the possibility of studying simplified subsystems.
2. Finance is an empirical science, but the ability to conduct experiments in finance is limited when compared with the experimental facilities built in the physical sciences.
3. Finance does not study laws of nature, but it studies a human artifact that is subject to change due to human decisions.
4. Finance systems are self-reflecting in the sense that the knowledge accumulated on the system changes the system itself.

None of the above four points is in itself an objection to the scientific study of finance as a mathematical science. However, it should be clear that the methods of scientific investigations and the findings of finance might be conceptually different from those of the physical sciences. It would probably be a mistake to expect in finance the same type of generalized axiomatic laws that we find in physics.

One of the major sources of the progress made by physics is due to the ability to isolate elementary subsystems, to come out with laws that apply to these subsystems, and then to recover macroscopic laws by a mathematical process. For example, the study of mechanics was greatly simplified by the study of the material point, a subsystem without structure identified by a small number of continuous variables. After identifying the laws that govern the motion of a material point, the motion of any physical body can be recovered by a process of mathematical integration. Simplifications of this type allow one to both simplify the mathematics and to perform empirical tests in a simplified environment.

In financial economics, however, we cannot study idealized subsystems because we cannot identify subsystems with a simplified behavior. This is not to say that attempts have not been made. Drawing on the principles of microeconomics, financial economics attempts to study the behavior of individuals as the elementary units of the financial system. The real problem, however, is that the study of individuals as economic "atoms" cannot produce simple laws because it is the study of the human financial decision-making process, which is very complex in itself. In addition, we cannot perform experiments. Instead, we have to rely on how the only financial system we know develops

itself.

Note that the possibility to study elementary subsystems does not coincide with the existence of fundamental laws. For example, consider the Schrödinger equation of quantum mechanics formulated in 1926 by the physicist Erwin Schrödinger. The equation is a partial differential equation describing how in some physical system a quantum state evolves over time. Although the Schrödinger equation is indeed a fundamental law, it applies to any system and not only to microscopic entities. Fundamental laws are not necessarily microscopic laws. We might be able to find fundamental laws of finance even if we are unable to isolate elementary subsystems.

There is a strong connection between fundamental laws and the ability to make experiments. By their nature, fundamental laws are very general and can be applied, albeit after difficult mathematical manipulations, to any phenomena. Therefore, after discovering a fundamental law it is generally possible to design experiments specific to test that same law. In many instances in the history of physics, crucial experiments have suggested rejection of a theory in favor of a new competing theory. However, in finance the ability to conduct experiments is limited though important research in the field has been carried on. In the 1970s, Daniel Kahneman and Amos Tversky performed groundbreaking research on cognitive biases in decision making. Vernon Smith studied different types of market organization, in particular auctions. This type of research has changed the perspective on finance as an empirical science. Still, we cannot make a close parallel between experimental finance and experimental physics where we can design experiments to decide between theories.

Perhaps the deepest difference between finance and physics is the fact that finance studies a human artifact which is subject to change in function of human decisions. Physics aims at discovering fundamental physical laws while finance determines laws that apply to a specific artifact. The level of generality of finance is intrinsically lower than that of physics. In addition, financial systems tend to change in function of the knowledge accumulated so that the object of inquiry is not stable.

As a result of all the above, it is unlikely that the kind of mathematics used in physics is appropriate to the study of financial theories. For example, we cannot expect to find any simple law that might be expressed with a closed formula. Hence, empirical testing cannot be done by comparing the results of closed-form solutions with experiments but more likely by comparing the results of long calculations. Thus the mathematical description of financial systems was delayed until researchers in finance had high-performance computers to perform the requisite large number of calculations. Nor can we expect a great level of accuracy in our descriptions of financial phenomena. If we want to compare finance with the natural sciences, we have to compare our knowledge of finance with our knowledge of the laws that govern macrosystems. While physicists have been able to determine extremely precise laws that govern subsystems such as atoms, their ability to predict macroscopic phenomena such as earthquakes or weather remains quite limited. Parallels between finance and the natural sciences are to be found more in the theory of complex systems than in fundamental physics.

In this book, special emphasis has been put on describing concepts and mathematical techniques, leaving aside lengthy demonstrations, which, while the substance of mathematics, are of limited interest to the practitioner and student of financial economics. From the practitioner's point of view, what is important is to have a firm grasp of the concepts and techniques so as to understand their appropriate application. There is no prerequisite mathematical knowledge for reading this book: all mathematical concepts used in the book are explained, starting from ordinary calculus and matrix algebra. It is, however, a demanding book given the breadth and depth of concepts covered. Each chapter begins with a brief description of how the tool it covers is used in finance, which is the

followed by the learning objectives for the chapter. Each chapter concludes with its key points.

In writing this book, special attention was given to bridging the gap between the intuition of the practitioner and academic mathematical analysis. Often there are simple compelling reasons for adopting sophisticated concepts and techniques that are obscured by mathematical details. That said, whenever possible, we tried to give the reader an understanding of the reasoning behind the concepts. The book has many examples of how quantitative analysis is used in the practice of asset management.

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<sup>1</sup> E. Wigner, “The Unreasonable Effectiveness of Mathematics in the Natural Sciences,”  
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# CHAPTER 1

## Basic Concepts

### Sets, Functions, and Variables

In mathematics, sets, functions, and variables are three fundamental concepts. First, a **set** is a well-defined collection of objects. A set is a gathering together into a whole of definite, distinct objects of our perception, which are called elements of the set. Sets are one of the most fundamental concepts in mathematics. Set theory is seen as the foundation from which virtually all of mathematics can be derived. For example, structures in abstract algebra, such as groups, fields, and rings, are sets closed under one or more operations. One of the main applications of set theory is constructing relations. Second, a **function** is a relation between a set of inputs and a set of permissible outputs with the property that each input is related to exactly one output. Functions are the central objects of investigation in most fields of modern mathematics. There are many ways to describe or represent a function. Some functions may be defined by a formula or algorithm that tells how to compute the output for a given input. Others are given by a picture, called the **graph of the function**. A function can be described through its relationship with other functions, for example, as an inverse function or as a solution of a differential equation. Finally, a **variable** is a value that may change within the scope of a given problem or set of operations. In contrast, a **constant** is a value that remains unchanged, though often unknown or undetermined. Variables are further distinguished as being either a dependent variable or an independent variable. Independent variables are regarded as inputs to a system and may take on different values freely. Dependent variables are those values that change as a consequence of changes in other values in the system.

The concepts of sets, functions, and variables are fundamental to many areas of finance and its applications. Starting with the mean-variance portfolio theory of Harry Markowitz in 1952, then the capital asset pricing model of William Sharpe in 1964, the option pricing model of Fischer Black and Myron Scholes in 1973, and the more recent developments in financial econometrics, financial risk management and asset pricing, financial economists constantly use the concepts of sets, functions, and variables. In this chapter we discuss these concepts.

#### What you will learn after reading this chapter:

- The notion of sets and set operations
- How to define empty sets, union of sets, and intersection of sets.
- The elementary properties of sets.
- How to describe the dynamics of quantitative phenomena.
- The concepts of distance and density of points.
- How to define and use functions and variables.

## INTRODUCTION

In this chapter we discuss three basic concepts used throughout this book: sets, functions, and variables. These concepts are used in financial economics, financial modeling, and financial econometrics.

## SETS AND SET OPERATIONS

The basic concept in calculus and in probability theory is that of a **set**. A set is a collection of objects called **elements**. The notions of both elements and set should be considered primitive. Following

common convention, let's denote sets with capital Latin or Greek letters:  $A, B, C, \Omega \dots$  and elements with small Latin or Greek letters:  $a, b, \omega$ . Let's then consider collections of sets. In this context, a set is regarded as an element at a higher level of aggregation. In some instances, it might be useful to use different alphabets to distinguish between sets and collections of sets.<sup>1</sup>

## Proper Subsets

An element  $a$  of a set  $A$  is said to belong to the set  $A$  written as  $a \in A$ . If every element that belongs to a set  $A$  also belongs to a set  $B$ , we say that  $A$  is contained in  $B$  and write:  $A \subset B$ . We will distinguish whether  $A$  is a **proper subset** of  $B$  (i.e., whether there is at least one element that belongs to  $B$  but not to  $A$ ) or if the two sets might eventually coincide. In the latter case we write  $A \subseteq B$ .

In the United States there are indexes that are constructed based on the price of a subset of common stocks from the universe of all common stock in the country. There are three types of common stock (equity) indexes:

1. Produced by stock exchanges based on all stocks traded on the particular exchanges (the most well known being the New York Stock Exchange Composite Index).
2. Produced by organizations that subjectively select the stocks included in the index (the most popular being the Standard & Poor's 500).
3. Produced by organizations where the selection process is based on an objective measure such as market capitalization.

The Russell equity indexes, produced by Frank Russell Company, are examples of the third type of index. The Russell 3000 Index includes the 3,000 largest U.S. companies based on total market capitalization. It represents approximately 98% of the investable U.S. equity market. The Russell 1000 Index includes 1,000 of the largest companies in the Russell 3000 Index while the Russell 2000 Index includes the 2,000 smallest companies in the Russell 3000 Index. The Russell Top 200 Index includes the 200 largest companies in the Russell 1000 Index and the Russell Midcap Index includes the 800 smallest companies in the Russell 1000 Index. None of the indexes include non-U.S. common stocks.

Let us introduce the notation:

$A$  = all companies in the United States that have issued common stock

$I_{3000}$  = companies included in the Russell 3000 Index

$I_{1000}$  = companies included in the Russell 1000 Index

$I_{2000}$  = companies included in the Russell 2000 Index

$I_{\text{Top200}}$  = companies included in the Russell Top 200 Index

$I_{\text{Midcap}}$  = companies included in the Russell Midcap 200 Index

We can then write the following:

|                                                                                                                                    |                                                                                                                                                         |
|------------------------------------------------------------------------------------------------------------------------------------|---------------------------------------------------------------------------------------------------------------------------------------------------------|
| $I_{3000} \subset A$                                                                                                               | (every company that is contained in the Russell 3000 Index is contained in the set of all companies in the United States that have issued common stock) |
| $I_{1000} \subset I_{3000}$                                                                                                        | (the largest 1,000 companies contained in the Russell 1000 Index are contained in the Russell 3000 Index)                                               |
| $I_{\text{Midcap}} \subset I_{1000}$                                                                                               | (the 800 smallest companies in the Russell Midcap Index are contained in the Russell 1000 Index)                                                        |
| $I_{\text{Top200}} \subset I_{1000} \subset I_{3000} \subset A$<br>$I_{\text{Midcap}} \subset I_{1000} \subset I_{3000} \subset A$ |                                                                                                                                                         |

Throughout this book we will make use of the convenient logic symbols  $\forall$  and  $\exists$  that mean respectively, “for any element” and “an element exists such that.” We will also use the symbol  $\Rightarrow$  that means “implies.” For instance, if  $A$  is a set of real numbers and  $a \in A$ , the notation  $\forall a: a < x$  means “for any number  $a$  smaller than  $x$ ” and  $\exists a: a < x$  means “there exists a number  $a$  smaller than  $x$ .”

## Empty Sets

Given a subset  $B$  of a set  $A$ , the complement of  $B$  with respect to  $A$  written as  $B^C$  is formed by all elements of  $A$  that do not belong to  $B$ . It is useful to consider sets that do not contain any elements called **empty sets**. The empty set is usually denoted by  $\emptyset$ . For example, stocks with negative price form an empty set.

## Union of Sets

Given two sets  $A$  and  $B$ , their **union** is formed by all elements that belong to either  $A$  or  $B$ . This is written as  $C = A \cup B$ . For example,

|                                                       |                                                                                                                                                                  |
|-------------------------------------------------------|------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| $I_{1000} \cup I_{2000} = I_{3000}$                   | (the union of the companies contained in the Russell 1000 Index and the Russell 2000 Index is the set of all companies contained in the Russell 3000 Index)      |
| $I_{\text{Midcap}} \cup I_{\text{Top200}} = I_{1000}$ | (the union of the companies contained in the Russell Midcap Index and the Russell Top 200 Index is the set of all companies contained in the Russell 1000 Index) |

Let  $I_{\text{Long lived}}$  be those stocks that existed in the last 30 years.

## Intersection of Sets

Given two sets  $A$  and  $B$ , their **intersection** is formed by all elements that belong to both  $A$  and  $B$ . This is written as  $C = A \cap B$ . For example, let

$I_{\text{S\&P}}$  = companies included in the S&P 500 Index

The S&P 500 is a stock market index that includes 500 widely held common stocks representing about 77% of the New York Stock Exchange market capitalization. (**Market capitalization** for a company is the product of the market value of a share and the number of shares outstanding.) Call  $I_{\text{Long lived}}$  the set of stocks that existed in the last 30 years. Then

|                                                  |                                                                                |
|--------------------------------------------------|--------------------------------------------------------------------------------|
| $I_{\text{S\&P}} \cap I_{\text{Long lived}} = C$ | (the stocks contained in the S&P 500 Index that existed for the last 30 years) |
|--------------------------------------------------|--------------------------------------------------------------------------------|

We can also write:

|                                      |                                                                                                                                                 |
|--------------------------------------|-------------------------------------------------------------------------------------------------------------------------------------------------|
| $I_{1000} \cap I_{2000} = \emptyset$ | (companies included in both the Russell 2000 and the Russell 1000 Index is the empty set since there are no companies that are in both indexes) |
|--------------------------------------|-------------------------------------------------------------------------------------------------------------------------------------------------|

## Elementary Properties of Sets

Suppose that the set  $\Omega$  includes all elements that we are presently considering (i.e., that it is the total set). Three elementary properties of sets are given below:

*Property 1.* The complement of the total set is the empty set and the complement of the empty set is the total set:

$$\Omega^C = \emptyset, \emptyset^C = \Omega$$

*Property 2.* If  $A, B, C$  are subsets of  $\Omega$ , then the distribution properties of union and intersection hold:

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$

$$\neg A \cap (B \cup C) = (\neg A \cap B) \cup (\neg A \cap C)$$


---

**Property 3.** The complement of the union is the intersection of the complements and the complement of the intersection is the union of the complements:

$$(B \cup C)^C = B^C \cap C^C$$

$$(B \cap C)^C = B^C \cup C^C$$

## DISTANCES AND QUANTITIES

Calculus describes the dynamics of quantitative phenomena. This calls for equipping sets with metric that defines distances between elements. Though many results of calculus can be derived in abstract metric spaces, standard calculus deals with sets of ***n*-tuples** of real numbers. In a quantitative framework, real numbers represent the result of observations (or measurements) in a simple and natural way.

### *n*-tuples

An *n*-tuple, also called an ***n*-dimensional vector**, includes *n* components:  $(a_1, a_2, \dots, a_n)$ . The set of all *n*-tuples of real numbers is denoted by  $R^n$ . The  $R$  stands for real numbers.

For example, suppose the monthly rates of return on a hedge fund portfolio in 2011 are as shown in [Table 1.1](#) with the actual return for the S&P 500 (the benchmark index for the hedge fund portfolio manager).<sup>2</sup>

**TABLE 1.1** Monthly Returns for the Hedge Fund Composite and S&P 500 Indexes

| Month     | Hedge Fund Portfolio | S&P 500 |
|-----------|----------------------|---------|
| January   | 0.41%                | 2.26%   |
| February  | 1.23%                | 3.20%   |
| March     | 0.06%                | -0.10%  |
| April     | 1.48%                | 2.85%   |
| May       | -1.20%               | -1.35%  |
| June      | -1.18%               | -1.83%  |
| July      | 0.23%                | -2.15%  |
| August    | -3.21%               | -5.68%  |
| September | -3.89%               | -7.18%  |
| October   | 2.67%                | 10.77%  |
| November  | -1.29%               | -0.51%  |
| December  | -0.43%               | 0.85%   |

Then the monthly returns,  $r_{\text{port}}$ , for the hedge fund portfolio can be written as a 12-tuple and has the following 12 components:

$$r_{\text{port}} = \begin{bmatrix} 0.41\%, 1.23\%, 0.06\%, 1.48\%, -1.20\%, -1.18\% \\ 0.23\%, -3.21\%, -3.89\%, 2.67\%, -1.29\%, -0.43\% \end{bmatrix}$$

Similarly, the return  $r_{\text{S\&P}}$  on the S&P 500 can be expressed as a 12-tuple as follows:

$$r_{\text{S\&P}} = \begin{bmatrix} 2.26\%, 3.20\%, -0.10\%, 2.85\%, -1.35\%, -1.83\% \\ -2.15\%, -5.68\%, -7.18\%, 10.77\%, -0.51\%, 0.85\% \end{bmatrix}$$

One can perform standard operations on  $n$ -tuples. For example, consider the hedge fund portfolio returns in the two 12-tuples. The 12-tuple that expresses the deviation of the hedge fund portfolio performance from the benchmark S&P 500 index is computed by subtracting from each component of the return 12-tuple from the corresponding return on the S&P 500. That is,

$$\begin{aligned} r_{\text{port}} - r_{\text{S\&P}} &= \begin{bmatrix} 0.41\%, 1.23\%, 0.06\%, 1.48\%, -1.20\%, -1.18\% \\ 0.23\%, -3.21\%, -3.89\%, 2.67\%, -1.29\%, -0.43\% \end{bmatrix} \\ &\quad - \begin{bmatrix} 2.26\%, 3.20\%, -0.10\%, 2.85\%, -1.35\%, -1.83\% \\ -2.15\%, -5.68\%, -7.18\%, 10.77\%, -0.51\%, 0.85\% \end{bmatrix} \\ &= \begin{bmatrix} -1.86\%, -1.96\%, 0.17\%, -1.37\%, 0.15\%, 0.65\% \\ 2.37\%, 2.46\%, 3.29\%, -8.10\%, -0.78\%, -1.29\% \end{bmatrix} \end{aligned}$$

It is the resulting 12-tuple that is used to compute the **tracking error** of a portfolio—the standard deviation of the variation of the portfolio’s return from its benchmark index’s return.

Coming back to the portfolio return, one can compute a logarithmic return for each month by adding 1 to each component of the 12-tuple and then taking the natural logarithm of each component. One can then obtain a geometric average, called the **geometric return**, by multiplying each component of the resulting vector and taking the 12th root.

## Distance

Consider the real line  $R^1$  (i.e., the set of real numbers). Real numbers include rational numbers and irrational numbers. A **rational number** is one that can be expressed as a fraction,  $c/d$ , where  $c$  and  $d$  are integers and  $d \neq 0$ . An **irrational number** is one that cannot be expressed as a fraction. Three examples of irrational numbers are

$$\sqrt{2} \cong 1.4142136$$

Ratio between diameter and circumference

$$= \pi \cong 3.1415926535897932384626$$

Natural logarithm =  $e \cong 2.7182818284590452353602874713526$

On the real line, distance is simply the absolute value of the difference between two numbers  $|a - b|$  which also can be written as

$$\sqrt{(a - b)^2}$$

$R^n$  is equipped with a natural metric provided by the Euclidean distance between any two points

$$d[(a_1, a_2, \dots, a_n), (b_1, b_2, \dots, b_n)] = \sqrt{\sum (a_i - b_i)^2}$$

Given a set of numbers  $A$ , we can define the least upper bound of the set. This is the smallest number  $s$  such that no number contained in the set exceeds  $s$ . The quantity  $s$  is called the **supremum** and written as  $s = \sup A$ . More formally, the supremum is that number, if it exists, that satisfies the following properties:

$$\forall a : a \in A, s \geq a$$

$$\forall \varepsilon > 0, \exists a : s - a \leq \varepsilon$$

where  $\varepsilon$  is any real positive number. The supremum need not belong to the set  $A$ . If it does, it is called the **maximum**.

Similarly, **infimum** is the greatest lower bound of a set  $A$ , defined as the greatest number  $s$  such that no number contained in the set is less than  $s$ . If infimum belongs to the set it is called the **minimum**.

# Density of Points

A key concept of set theory with a fundamental bearing on calculus is that of **density of points**. In fact, in financial economics we distinguish between discrete and continuous quantities. **Discrete quantities** have the property that admissible values are separated by finite distances. **Continuous quantities** are such that one might go from one to any of two possible values passing through every possible intermediate value. For instance, the passing of time between two dates is considered to occupy every possible instant without any gap.

The fundamental continuum is the set of real numbers. A **continuum** can be defined as any set that can be placed in a one-to-one relationship with the set of real numbers. Any continuum is an **infinite non-countable set**; a proper subset of a continuum can be a continuum. It can be demonstrated that any finite interval is a continuum as it can be placed in a one-to-one relationship with the set of all real numbers.

The intuition of a continuum can be misleading. To appreciate this, consider that the set of all rational numbers (i.e., the set of all fractions with integer numerator and denominator) has a dense ordering, that is, has the property that given any two different rational numbers  $a, b$  with  $a < b$ , there are infinite other rational numbers in between. However, rational numbers have the cardinality of the natural numbers. That is to say rational numbers can be put into a one-to-one relationship with natural numbers. This can be seen using a clever construction that we owe to the seventeenth-century Swiss mathematician Jacob Bernoulli.

Using Bernoulli's construction, we can represent rational numbers as fractions of natural numbers arranged in an infinite two-dimensional table in which columns grow with the denominators and rows grow with the numerators. A one-to-one relationship with the natural numbers can be established following the path: (1, 1) (1, 2) (2, 1) (3, 1) (2, 2) (1, 3) (1, 4) (2, 3) (3, 2) (4, 1) and so on (see [Table 1.2](#)).

**TABLE 1.2** Bernoulli's Construction to Enumerate Rational Numbers

|     |     |     |     |
|-----|-----|-----|-----|
| 1/1 | 1/2 | 1/3 | 1/4 |
| 2/1 | 2/2 | 2/3 | 2/4 |
| 3/1 | 3/2 | 3/3 | 3/4 |
| 4/1 | 4/2 | 4/3 | 4/4 |

Bernoulli thus demonstrated that there are as many rational numbers as there are natural numbers. Though the set of rational numbers has a dense ordering, rational numbers do not form a continuum because they cannot be put in a one-to-one correspondence with real numbers.

Given a subset  $A$  of  $R^n$ , a point  $a \in A$  is said to be an **accumulation point** if any sphere centered in  $a$  contains an infinite number of points that belong to  $A$ . A set is said to be "closed" if it contains all of its own accumulation points and "open" if it does not.

## FUNCTIONS

The mathematical notion of a function translates the intuitive notion of a relationship between two quantities. For example, the price of a security is a function of time: to each instant of time corresponds a price of that security.

Formally, a **function**  $f$  is a mapping of the elements of a set  $A$  into the elements of a set  $B$ . The set  $A$  is called the **domain** of the function. The subset  $R = f(A) \subseteq B$  of all elements of  $B$  that are the mapping

of some element in  $A$  is called the **range**  $R$  of the function  $f$ .  $R$  might be a proper subset of  $B$  coincide with  $B$ .

The concept of function is general: the sets  $A$  and  $B$  might be any two sets, not necessarily sets of numbers. When the range of a function is a set of real numbers, the function is said to be a **real function** or a **real-valued function**.

Two or more elements of  $A$  might be mapped into the same element of  $B$ . Should this situation never occur, that is, if distinct elements of  $A$  are mapped into distinct elements of  $B$ , the function is called an **injection**. If a function is an injection and  $R = f(A) = B$ , then  $f$  represents a one-to-one relationship between  $A$  and  $B$ . In this case the function  $f$  is invertible and we can define the **inverse function**  $g = f^{-1}$  such that  $f(g(a)) = a$ .

Suppose that a function  $f$  assigns to each element  $x$  of set  $A$  some element  $y$  of set  $B$ . Suppose further that a function  $g$  assigns an element  $z$  of set  $C$  to each element  $y$  of set  $B$ . Combining functions  $f$  and  $g$ , an element  $z$  in set  $C$  corresponds to an element  $x$  in set  $A$ . This process results in a new function  $h$ , and that function  $h$  takes an element in set  $A$  and assigns it to set  $C$ . The function  $h$  is called the composite of functions  $g$  and  $f$ , or simply a **composite function**, and is denoted by  $h(x) = g[f(x)]$ .

## VARIABLES

In applications in finance, one usually deals with functions of numerical variables. Some distinctions are in order. A **variable** is a symbol that represents any element in a given set. For example, if we denote time with a variable  $t$ , the letter  $t$  represents any possible moment of time. **Numerical variables** are symbols that represent numbers. These numbers might, in turn, represent the elements of another set. They might be thought of as numerical indexes which are in a one-to-one relationship with the elements of a set. For example, if we represent time over a given interval with a variable  $t$ , the letter  $t$  represents any of the numbers in the given interval. Each of these numbers in turn represents an instant of time. These distinctions might look pedantic but they are important for the following two reasons.

First, we need to consider **numeraire** or units of measure. Suppose, for instance, that we represent the price  $P$  of a security as a function of time  $t$ :  $P = f(t)$ . The function  $f$  links two sets of numbers that represent the physical quantities price and time. If we change the time scale or the currency, the numerical function  $f$  will change accordingly though the abstract function that links time and price will remain unchanged.

Variables can be classified as qualitative or quantitative. Qualitative (or categorical) variables take on values that are names or labels. Examples of qualitative variables would include the color of a ball (e.g., red, green, blue) or a dummy variable (also known as an indicator variable) taking the values 0 or 1. Quantitative variables are numerical. They represent a measurable quantity. For example, when we speak of the population of a city, we are talking about the number of people in the city, which is a measurable attribute of the city. Therefore, population would be a quantitative variable.

Variables can also be classified as deterministic or random. In probability and statistics, a random variable, or stochastic variable, is a variable that can take on a set of possible different values, each with an associated probability. For example, when a coin is tossed 10 times, the random variable is the number of tails (or heads) that are noted.  $X$  can only take the values 0, 1, ..., 10, so in this example  $X$  is a discrete random variable. Variables might represent phenomena that evolve over time. A deterministic variable evolves according to fixed rules, for example an investment that earns a fixed



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