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Mod Two Homology and Cohomology

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Jean-Claude Hausmann

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Jean-Claude Hausmann
University of Geneva, Geneva, Switzerland

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1. Introduction

Jean-Claude Hausmann

(1) University of Geneva, Geneva, Switzerland

Jean-Claude Hausmann
Email: jean-claude.hausmann@unige.ch

Mod 2 homology first occurred in 1908 in a paper of Tietze [196] (see also [40, pp. 41–42]). Several results were first established using this mod 2 approach, like the linking number for submanifolds in $\mathbb{R}^n$ (see Sect. 5.4.4), as well as Alexander duality [7]. One argument in favor of the choice of the mod 2 homology was its simplicity, as J.W. Alexander says in his introduction: "The theory of connectivity [homology] may be approached from two different angles depending on whether or not the notion of *sense* [orientation] is developed and taken into consideration. We have adopted the second and somewhat simpler point of view in this discussion in order to condense the necessary preliminaries as much as possible. A treatment involving the idea of sense would be somewhat more complicated but would follow along much the same lines."

Besides being simpler than its integral counterpart, mod 2 homology sometimes gives new theorems. The first historical main example is the generalization of Poincaré duality to all closed manifolds, whether orientable or not, a result obtained by Veblen and Alexander in 1913 [200]. As a consequence, the Euler characteristic of a closed odd-dimensional manifold vanishes.

The discoveries of Stiefel-Whitney classes in 1936–1938 and of Steenrod squares in 1947–1950 gave mod 2 cohomology the status of a major tool in algebraic topology, providing for instance the theory of spin structures and Thom's work on the cobordism ring.

These notes are an introduction, at graduate student's level, of mod 2 (co)homology (there will be essentially no other). They include classical applications (Brouwer fixed point theorem, Poincaré duality, Borsuk-Ulam theorem, Smith theory, etc) and less classical ones (face spaces, topological complexity, equivariant Morse theory, etc). The cohomology of flag manifolds is treated in details, including for Grassmannians the relationship between Stiefel-Whitney classes and Schubert calculus. Some original applications are given in Chap. 10.

Our approach is different than that of classical textbooks, in which mod 2 (co)homology is just a particular case of (co)homology with arbitrary coefficients. Also, most authors start with a full account of homology before approaching cohomology. In these notes, mod 2 (co)homology is treated as a subject by itself and we start with cohomology and homology together from the beginning. The advantages of this approach are the following.

- The definition of a (co)chain is simple and intuitive: an (say, simplicial) *m-cochain* is a set of m-simplexes; an *m-chain* is a *finite* set of m-simplexes. The concept of cochain is simpler than that of chain (one less word in the definition …), more flexible and somehow more natural. We thus tend to consider cohomology as the main concept and homology as a (useful) tool for some arguments.

- Working with $\mathbb{Z}_2$ and its standard linear algebra is much simpler than working with $\mathbb{Z}$. For instance, the *Kronecker pairing* has an intuitive geometric interpretation occurring at the beginning which shows in an elementary way that cohomology is the dual of homology. Several computations, like the homology of surfaces, are quite easy and come early in the exposition. Also, the cohomology ring is *commutative*. The cup square $a \to a \smile a$ is a linear map and may be also non-trivial in odd degrees, leading to important invariants.
- The absence of sign and orientation considerations is an enormous technical simplification (even of importance in computer algorithms computing homology). With much lighter computations and technicalities, the ideas of proofs are more apparent.

We hope that these notes will be, for students and teachers, a complement or companion to textbooks like those of Hatcher [82] or Munkres [153]. From our teaching experience, starting with $\mathrm{mod}\ 2$ (co)homology and taking advantage of its above mentioned simplicity is a great help to grasp the ideas of the subject. The technical difficulties of signs and orientations for finer theories, like integral (co)homology, may then be introduced afterwards, as an adaptation of the more intuitive $\mathrm{mod}\ 2$ (co)homology.

Not in this book The following tools are not used in these notes.

- Augmented (co)chain complexes. The reduced cohomology $\tilde{H}^*(X)$ is defined as $\mathrm{coker}\,(H^*(pt) \to H^*(X))$ for the unique map $X \to pt$.
- Simplicial approximation.
- Spectral sequences (except in the proof of Proposition 7.2.17).

Also, we do not use advanced homotopy tools, like spectra, completions, etc. Because of this, some prominent problems using $\mathrm{mod}\ 2$ cohomology are only briefly surveyed, like the work by Adams on the Hopf-invariant-one problem (p. 353), the Sullivan's conjecture (pp. 240 and 353) and the Kervaire invariant (Sect. 10.6).

Prerequisites The reader is assumed to have some familiarity with the following subjects:

- general point set topology (compactness, connectedness, etc).
- elementary language of categories and functors.
- simple techniques of exact sequences, like the five lemma.
- elementary facts about fundamental groups, coverings and higher homotopy groups (not much used).
- elementary techniques of smooth manifolds.

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2. Simplicial (Co)homology

Jean-Claude Hausmann

(1) University of Geneva, Geneva, Switzerland

Jean-Claude Hausmann
Email: jean-claude.hausmann@unige.ch

Simplicial homology was invented by Poincaré in 1899 [162] and its $\mathrm{mod}\ 2$ version, presented in this chapter, was introduced in 1908 by Tietze [196]. It is the simplest homology theory to understand an for finite complexes, it may be computed algorithmically. The $\mathrm{mod}\ 2$ version permits rapid computations on easy but non-trivial examples, like spheres and surfaces (see Sect. 2.4).

Simplicial (co)homology is defined for a simplicial complex, but is an invariant of the homotopy type of its geometric realization (this result will be obtained in different ways using singular homology: see Sect. 3.6). The first section of this chapter introduces classical techniques of (abstract simplicial complexes. Since simplicial homology was the only existing (co)homology theory until th 1930s, simplicial complexes played a predominant role in algebraic topology during the first third of the 12th century (see the Introduction of Sect. 5.1). Later developments of (co)homology theories, defined directly for topological spaces, made this combinatorial approach less crucial. However, simplicial complexes remain an efficient way to construct topological spaces, also largely used in computer science.

2.1 Simplicial Complexes

In this section we fix notations and recall some classical facts about (abstract) simplicial complexes. For more details, see [179, Chap. 3].

A *simplicial complex* K consists of

- a set $V(K)$, the set of *vertices* of K.
- a set $\mathcal{S}(K)$ of finite non-empty subsets of $V(K)$ which is closed under inclusion: if $\sigma \in \mathcal{S}(K)$ and $\tau \subset \sigma$, then $\tau \in \mathcal{S}(K)$. We require that $\{v\} \in \mathcal{S}(K)$ for all $v \in V(K)$.

An element σ of $\mathcal{S}(K)$ is called a *simplex* of K ("simplexes" and "simplices" are admitted as plural of "simplex"; we shall use "simplexes", in analogy with "complexes"). If $\sharp(\sigma) = m+1$, we say that σ is of *dimension* m or that σ is an m *-simplex*. The set of m-simplexes of K is denoted by $\mathcal{S}_m(K)$. The set $\mathcal{S}_0(K)$ of 0-simplexes is in bijection with $V(K)$, and we usually identify $v \in V(K)$ wit $\{v\} \in \mathcal{S}_0(K)$. We say that K is of *dimension* $\leq n$ if $\mathcal{S}_m(K) = \emptyset$ for $m > n$, and that K is of *dimension* (or n *-dimensional*) it it is of dimension $\leq n$ but not of dimension $\leq n-1$. A simplicial complex of dimension ≤ 1 is called a *simplicial graph*. A simplicial complex K is called *finite* if $V(K)$ is a finite set.

If $\sigma \in S(K)$ and $\tau \subset \sigma$, we say that τ is a *face* of σ. As $S(K)$ is closed under inclusion, it is determined by it subset $S_{max}(K)$ of *maximal* simplexes (if K is finite dimensional). A *subcomplex* L of K is a simplicial complex such that $V(L) \subset V(K)$ and $S(L) \subset S(K)$. If $S \subset S(K)$ we denote by $\bar{S}$ the subcomplex generated by S, i.e. the smallest subcomplex of K such that $S \subset S(\bar{S})$. The m *-skeleton* K^m of K is the subcomplex of K generated by the union of $S_k(K)$ for $k \leq m$.

Let $\sigma \in S(K)$. We denote by $\bar{\sigma}$ the subcomplex of K formed by σ and all its faces ($\overline{\{\sigma\}}$ in the above notation). The subcomplex $\dot{\sigma}$ of $\bar{\sigma}$ generated by the proper faces of σ is called the *boundary of* σ.

2.1.1

Geometric realization. The *geometric realization* $|K|$ of a simplicial complex K is, as a set, defined by

$$|K| := \{\mu : V(K) \to [0,1] \mid \textstyle\sum_{v \in V(K)} \mu(v) = 1 \text{ and } \mu^{-1}((0,1]) \in S(K)\}$$

We can thus see $|K|$ as the set of probability measures on $V(K)$ which are supported by the simplexes (this language is just used for comments and only in this section). There is a distance on $|K|$ defined by

$$d(\mu, \nu) = \sqrt{\sum_{v \in V(K)} |\mu(v) - \nu(v)|^2}$$

which defines the *metric topology* on $|K|$. The set $|K|$ with the metric topology is denoted by $|K|_d$. For instance, if $\sigma \in S_m(K)$, then $|\bar{\sigma}|_d$ is isometric to the standard Euclidean simplex $\Delta^m = \{(x_0, \dots, x_m) \in \mathbf{R}^{m+1} \mid x_i \geq 0 \text{ and } \sum x_i = 1\}$.

However, a more used topology for $|K|$ is the *weak topology*, for which $A \subset |K|$ is closed if and only if $A \cap |\bar{\sigma}|_d$ is closed in $|\bar{\sigma}|_d$ for all $\sigma \in S(K)$. The notation $|K|$ stands for the set $|K|$ endowed with the weak topology. A map f from $|K|$ to a topological space X is then continuous if and only if its restriction to $|\bar{\sigma}|_d$ is continuous for each $\sigma \in S(K)$. In particular, the identity $|K| \to |K|_d$ is continuous, which implies that $|K|$ is Hausdorff. The weak and the metric topology coincide if and only if K is locally finite, that is each vertex is contained in a finite number of simplexes. When K is not locally finite, $|K|$ is not metrizable (see e.g. [179, Theorem 3.2.8]).

When a simplicial complex K is locally finite, has countably many vertices and is finite dimensional, it admits a *Euclidean realization*, i.e. an embedding of $|K|$ into some Euclidean space $\mathbf{R}^N$ which is piecewise affine. A map $f\colon |K| \to \mathbf{R}^N$ is *piecewise affine* if, for each $\sigma \in S(K)$, the restriction of f to $|\bar{\sigma}|$ is an affine map. Thus, for each simplex σ, the image of $|\bar{\sigma}|$ is an affine simplex of $\mathbb{R}^N$. If $\dim K \leq n$, such a realization exists in $\mathbb{R}^{2n+1}$ (see e.g. [179, Theorem 3.3.9]).

If $\sigma \in \mathcal{S}(K)$ then $|\bar\sigma| \subset |K|$. We call $|\bar\sigma|$ the *geometric simplex* associated to σ. Its *boundary* is $|\dot\sigma|$. The space $|\bar\sigma| - |\dot\sigma|$ is the *geometric open simplex* associated to σ. Observe that $|K|$ is the disjoint union of its geometric open simplexes.

There is a natural injection $i : V(K) \hookrightarrow |K|$ sending v to the Dirac measure with value 1 on v. We usually identify v with $i(v)$, seeing a simplex v as a point of $|K|$ (a geometric vertex). In this way, a point $\mu \in |K|$ may be expressed as a convex combination of (geometric) vertices:

$$\mu = \sum_{v \in V(K)} \mu(v) v \,. \quad (2.1.$$

2.1.2

Let K and L be simplicial complexes. Their *join* is the simplicial complex $K * L$ defined by

(1) $V(K * L) = V(K) \dot\cup V(L)$,

(2) $\mathcal{S}(K * L) = \mathcal{S}(K) \cup \mathcal{S}(L) \cup \{\sigma \cup \tau \mid \sigma \in \mathcal{S}(K) \text{ and } \tau \in \mathcal{S}(L)\}$.

Observe that, if $\sigma \in \mathcal{S}_i(K)$ and $\tau \in \mathcal{S}_k(L)$, then $\sigma \cup \tau \in \mathcal{S}_{i+k+1}(K * L)$. Also, $\overline{\sigma \cup \tau} = \bar\sigma * \bar\tau$ and $|K * L|$ the topological join of $|K|$ and $|L|$ (see p. 171).

2.1.3

Stars, links, etc. Let K be a simplicial complex and $\sigma \in \mathcal{S}(K)$. The *star* $\mathrm{St}(\sigma)$ *of* σ is the subcomplex of K generated by all the simplexes containing σ. The *link* $\mathrm{Lk}(\sigma)$ *of* σ is the subcomplex of K formed by the simplexes $\tau \subset \mathcal{S}(K)$ such that $\tau \cap \sigma = \emptyset$ and $\tau \cup \sigma \subset \mathcal{S}(K)$. Thus, $\mathrm{Lk}(\sigma)$ is a subcomplex of $\mathrm{St}(\sigma)$ and

$$\mathrm{St}(\sigma) = \bar\sigma * \mathrm{Lk}(\sigma)$$

More generally, if L is a subcomplex of K, the *star* $\mathrm{St}(L)$ *of* L is the subcomplex of K generated by all the simplexes containing a simplex of L. The *link* $\mathrm{Lk}(L)$ *of* L is the subcomplex of K formed by the simplexes $\tau \in \mathcal{S}(\mathrm{St}(L)) - \mathcal{S}(L)$. One has $\mathrm{St}(L) = L * \mathrm{Lk}(L)$. The *open star* $\mathrm{Ost}(L)$ of L is the open neighbourhood of $|L|$ in $|K|$ defined by

$$\mathrm{Ost}(L) = \{\mu \in |K| \mid \mu(v) > 0 \text{ if } v \in V(L)\}$$

This is the interior of $|\mathrm{St}(L)|$ in $|K|$.

2.1.4

Simplicial maps. Let K and L be two simplicial complexes. A *simplicial map* $f\colon K\to L$ is a map $f\colon V(K)\to V(L)$ such that $f(\sigma)\in S(L)$ if $\sigma\in S(K)$, i.e. the image of a simplex of K is a simplex of L. Simplicial complexes and simplicial maps form a category, the *simplicial category*, denoted by Simp

A simplicial map $f\colon K\to L$ induces a continuous map $|f|\colon |K|\to |L|$ defined, for $w\in V(L)$, by

$$|f|(\mu)(w)=\sum_{v\in f^{-1}(w)}\mu(v)$$

In other words, $|f|(\mu)$ is the pushforward of the probability measure μ on $|L|$. The geometric realization is thus a covariant functor from the simplicial category Simp to the topological category Top of topological spaces and continuous maps.

2.1.5

Components. Let K be a simplicial complex. We define an equivalence relation on $V(K)$ by saying that $v\sim v'$ if there exists $x_0,\dots,x_m\in V(K)$ with $x_0=v$, $x_m=v'$ and $\{x_i,x_{i+1}\}\in S(K)$. A maximal subcomplex L of K such that $V(L)$ is an equivalence class is called a *component* of K. The set of components of K is denoted by $\pi_0(K)$. As the vertices of a simplex are all equivalent, K is the disjoint union of its components and $\pi_0(K)$ is in bijection with $V(K)/\sim$. The relationship with $\pi_0(|K|)$, the set of (path)-components of the topological space $|K|$, is the following.

Lemma 2.1.6

The natural injection $j\colon V(K)\to|K|$ descends to a bijection $\bar{j}\colon \pi_0(K)\xrightarrow{\approx}\pi_0(|K|)$.

Proof

The definition of the relation $\sim$ makes clear that j descends to a map $\bar{j}\colon\pi_0(K)\to\pi_0(|K|)$. Any point of $|K|$ is joinable by a continuous path to some vertex $j(v)$. Hence, $\bar{j}$ is surjective. To check the injectivity of $\bar{j}$, let $v,v'\in V(K)$ with $\bar{j}(v)=\bar{j}(v')$. There exists then a continuous path $c\colon[0,1]\to|K|$ with $c(0)=j(v)$ and $c(1)=j(v')$. Consider the open cover $\{\mathrm{Ost}(w)\mid w\in V(K)\}$ of $|K|$. By compactness of $[0,1]$, there exists $n\in\mathbb{N}$ and vertices $v_0,\dots,v_{n-1}\in V(K)$ such that $c([k/n,(k+1)/n])\subset\mathrm{Ost}(v_k)$ for all $k=0,\dots,n-1$. As $c(0)=j(v)$ and $c(1)=j(v')$, one deduces that $v_0=v$ and $v_{n-1}=v'$. For $0<k\le n-1$ one has $c(k/n)\in\mathrm{Ost}(v_{k-1})\cap\mathrm{Ost}(v_k)$. This implies that $\{v_{k-1},v_k\}\in S(K)$ for all $k=1,\dots,n-1$, proving that $v\sim v'$. □

A simplicial complex is called *connected* if it is either empty or has one component. Note that $|K|$ is locally path-connected for any simplicial complex K. Indeed, any point has a neighborhood of the form $|\mathrm{St}(v)|$ for some vertex v, and $|\mathrm{St}(v)|$ path-connected. Therefore, $|K|$ is path-connected if and only if $|K|$ is connected. Using Lemma 2.1.6, this proves the following lemma.

Lemma 2.1.7

Let K be a simplicial complex. Then K is connected if and only if $|K|$ is a connected space.

Finally, we note the functoriality of π_0. Let $f\colon K \to L$ be a simplicial map. If $v \sim v'$ for $v, v' \in V(K)$, then $f(v) \sim f(v')$, so f descends to a map $\pi_0 f\colon \pi_0(K) \to \pi_0(L)$. If $f\colon K \to L$ and $g\colon L \to M$ are two simplicial maps, then $\pi_0(g \circ f) = \pi_0 g \circ \pi_0 f$. Also, $\pi_0 \mathrm{id}_K = \mathrm{id}_{\pi_0(K)}$. Thus, π_0 is a covariant functor from the simplicial category **Simp** to the category **Set** of sets and maps.

2.1.8

Simplicial order. A *simplicial order* on a simplicial complex L is a partial order $\leq$ on $V(L)$ such that each simplex is totally ordered. For example, a total order on $V(L)$, as in examples where vertices are labeled by integers, is a simplicial order. A simplicial order always exists, as a consequence of the well-ordering theorem.

2.1.9

Triangulations. A *triangulation* of a topological space X is a homeomorphism $h\colon |K| \to X$, where K is a simplicial complex. A topological space is *triangulable* if it admits a triangulation. It will be useful to have a good process to triangulate some subspaces of $\mathbb{R}^m$. A compact subspace A of $\mathbb{R}^m$ is a *convex cell* if it is the set of solutions of families of affine equations and inequalities

$$f_i(x) = 0,\ i = 1, \dots, r \quad \text{and} \quad g_j(x) \geq 0,\ j = 1, \dots, s.$$

A *face* B of A is a convex cell obtained by replacing some of the inequalities $g_j \geq 0$ by the equations $g_j = 0$. The *dimension* of B is the dimension of the smallest affine subspace of $\mathbb{R}^m$ containing B. A *vertex* of A is a cell of dimension 0. By induction on the dimension, one proves that a convex cell is the convex hull of its vertices (see e.g. [138, Theorem 5.2.2]).

A *convex-cell complex* P is a finite union of convex cells in $\mathbb{R}^m$ such that:

(i) if A is a cell of P, so are the faces of A;

(ii) the intersection of two cells of P is a common face of each of them.

The *dimension* of P is the maximal dimension of a cell of P. The r-*skeleton* P^r is the subcomplex formed by the cells of dimension $\leq r$. The 0-skeleton coincides with the set $V(P)$ of *vertices* of P.

A partial order $\leq$ on $V(P)$ is an *affine order for* P if any subset $R \subset V(P)$ formed by affinely independent points is totally ordered. For instance, a total order on $V(P)$ is an affine order. The following lemma is a variant of [104, Lemma 1.4].

Lemma 2.1.10

Let P be a convex-cell complex. An affine order $\leq$ for P determines a triangulation $h_\leq\colon |L_\leq| \xrightarrow{\approx} P$, where $L_{\preceq}$ is a simplicial complex with $V(L_{\preceq}) = V(P)$. The homeomorphism $h_{\preceq}$ is piecewise affine an $\preceq$ is a simplicial order on $L_{\preceq}$.

Proof
The order $\preceq$ being chosen, we drop it from the notations. For each subcomplex Q of P, we shall construct a simplicial complex $L(Q)$ and a piecewise affine homeomorphism $h_Q\colon |L(Q)| \to Q$ such that,

(i) $V(L(Q)) = V(Q)$;

(ii) if $Q' \subset Q$, then $L(Q') \subset L(Q)$ and $h_{Q'}$ is the restriction of h_Q to $L(Q')$.

The case $Q = P$ will prove the lemma. The construction is by induction on the dimension of Q, settin $L(Q) = Q$ and $h_Q = \mathrm{id}$ if $\dim Q = 0$.

Suppose that $L(Q)$ and h_Q have been constructed, satisfying (i) and (ii) above, for each subcomplex Q of P of dimension $\leq k-1$. Let A be a k-cell of K with minimal vertex a. Then A is the topological cone, with cone-vertex a, of the union B of faces of A not containing a. The triangulation $h_B\colon |L(B)| \to |B|$ being constructed by induction hypothesis, define $L(A)$ to be the join $L(B) * \{a\}$ and h_A to be the unique piecewise affine extension of h_B. Observe that, if C is a face of A then h_C is the restriction to $L(C)$ of h_A. Therefore, this process may be used for each k-cell of P to construct $h_Q\colon |L(Q)| \to Q$ for each subcomplex Q of P with $\dim Q \leq k$. $\square$

2.1.11
Subdivisions. Let Z be a set and $\mathcal{A}$ be a family of subsets of Z. A simplicial complex L such that

(a) $V(L) \subset Z$;

(b) for each $\sigma \in S(L)$ there exists $A \in \mathcal{A}$ such that $\sigma \subset A$;

is called a $(Z, \mathcal{A})$ *-simplicial complex*, or a Z-simplicial complex *supported by* $\mathcal{A}$.

Let K be a simplicial complex. Let N be a $(|K|, GS(K))$-simplicial complex, where

$GS(K) = \{|\sigma| \mid \sigma \in S(K)\}$
is the family of geometric simplexes of K. A continuous map $f\colon |N| \to |K|$ is associated to N, defin

by

$$\jmath(\mu) = \sum_{w \in V(N)} \mu(w) w$$

In other word, $\jmath$ is the piecewise affine map sending each vertex of N to to the corresponding point in $|K|$. A *subdivision* of a simplicial complex K is a $(|K|, \mathcal{GS}(K))$-simplicial complex N for which the associated map $\jmath\colon |N| \to |K|$ is a homeomorphism (in other words, $\jmath$ is a triangulation of $|K|$).

Let N be a $(|K|, \mathcal{GS}(K))$-simplicial complex for a simplicial complex K. If L is a subcomplex of K, then

$$N_L = \{\sigma \in S(N) \mid \sigma \subset |L|\}$$

is a $(|L|, \mathcal{GS}(L))$-simplicial complex. Its associated map $\jmath_L\colon |N_L| \to |L|$ is the restriction of $\jmath$ to $|L|$. The following Lemma is useful to recognize a subdivision (compare [179, Chap. 3, Sect. 3, Theorem 4]).

Lemma 2.1.12
Let N be a $(|K|, \mathcal{GS}(K))$-simplicial complex. Then N is a subdivision of K if and only if, for each $\tau \in S(K)$, the simplicial complex $N_{\bar\tau}$ is finite and $\jmath_{\bar\tau}\colon |N_{\bar\tau}| \to |\tau|$ is bijective.

Proof
If N is a subdivision of K, then $\jmath_{\bar\tau}$ is bijective since $\jmath$ is a homeomorphism. Also, $|N_{\bar\tau}| = \jmath^{-1}(|\bar\tau|)$ is compact, so $N_{\bar\tau}$ is finite.

Conversely, The fact that $\jmath_{\bar\tau}$ is bijective for each $\tau \in S(K)$ implies that the continuous map $\jmath$ is bijective. If $N_{\bar\tau}$ is finite, then $\jmath_{\bar\tau}$ is a continuous bijection between compact spaces, hence a homeomorphism. This implies that the map $\jmath^{-1}$, restricted to each geometric simplex, is continuous. Therefore, $\jmath^{-1}$ is continuous since K is endowed with the weak topology. □

Seeing $V(K)$ as a subset of $|K|$, we get the following corollary.

Corollary 2.1.13
Let N be a subdivision of K. Then $V(K) \subset V(N)$.

A useful systematic subdivision process is the barycentric subdivision. Let $\sigma \in S_m(K)$ be an m-simplex of a simplicial complex K. The *barycenter* $\hat\sigma \in |K|$ of σ is defined by

$$\hat\sigma = \frac{1}{m+1} \sum_{v \in \sigma} v .$$

The *barycentric subdivision* K' of K is the $(|K|, \mathcal{GS}(K))$-simplicial complex where

- $V(K') = \{\hat\sigma \in |K| \mid \sigma \in S(K)\}$;
- $\{\hat\sigma_0, \dots, \hat\sigma_m\} \in S_m(K')$ whenever $\sigma_0 \subset \cdots \subset \sigma_m$ ($\sigma_i \neq \sigma_j$ if $i \neq j$).

Using Lemma 2.1.12, the reader can check that K' is a subdivision of K. Observe that the partial order "$\le$" defined by

$$\hat{\sigma} \le \hat{\tau} \iff \sigma \subset \tau \tag{2.1.}$$

is a simplicial order on K'.

2.2 Definitions of Simplicial (Co)homology

Let K be a simplicial complex. In this section, we give the definitions of the homology $H_*(K)$ and cohomology $H^*(K)$ of K under the various and peculiar forms available when the coefficients are in the field $\mathbb{Z}_2 = \{0, 1\}$.

Definition 2.2.1
(*subset definitions*)

(a) An m -*cochain* is a subset of $S_m(K)$.

(b) An m -*chain* is a finite subset of $S_m(K)$.

The set of m-cochains of K is denoted by $C^m(K)$ and that of m-chains by $C_m(K)$. By identifying $\sigma \in S_m(K)$ with the singleton $\{\sigma\}$, we see $S_m(K)$ as a subset of both $C_m(K)$ and $C^m(K)$. Each subset of $S_m(K)$ is determined by its characteristic function $\chi_A : S_m(K) \to \mathbb{Z}_2$, defined by

$$\chi_A(\sigma) = \begin{cases} 1 & \text{if } \sigma \in A \\ 0 & \text{otherwise} \end{cases}$$

This gives a bijection between subsets of $S_m(K)$ and functions from $S_m(K)$ to $\mathbb{Z}_2$. We see such a function as a colouring (0 = white and 1 = black). The following "colouring definition" is equivalent to the subset definition:

Definition 2.2.2
(*colouring definitions*)

(a) An m -*cochain* is a function $a : S_m(K) \to \mathbb{Z}_2$.

(b) An m -*chain* is a function $\alpha : S_m(K) \to \mathbb{Z}_2$ with finite support.

The colouring definition is used in low-dimensional graphical examples to draw (co)chains in black (bold lines for 1-(co)chains).

Definition 2.2.2 endow $C^m(K)$ and $C_m(K)$ with a structure of a $\mathbb{Z}_2$-vector space. The singletons provide a basis of $C_m(K)$, in bijection with $S_m(K)$. Thus, Definition 2.2.2b is equivalent to

Definition 2.2.3
$C_m(K)$ is the $\mathbb{Z}_2$-vector space with basis $S_m(K)$:

$$C_m(K) = \bigoplus_{\sigma \in S_m(K)} \mathbb{Z}_2\, \sigma$$

We shall pass from one of Definitions 2.2.1, 2.2.2 or 2.2.3 to another without notice; the context usually prevents ambiguity. We consider $C_*(K) = \oplus_{m\in\mathbb{N}} C_m(K)$ and $C^*(K) = \oplus_{m\in\mathbb{N}} C^m(K)$ as graded $\mathbb{Z}_2$-vector spaces. The convention $C_{-1}(K) = C^{-1}(K) = 0$ is useful.

We now define the *Kronecker pairing* on (co)chains

$$C^m(K) \times C_m(K) \xrightarrow{\langle\,,\,\rangle} \mathbb{Z}_2$$

by the equivalent formulae

$$\begin{aligned} \langle a, \alpha\rangle &= \sharp(a \cap \alpha) \pmod 2 && \text{using Definition 2.2.1a and b} \\ &= \textstyle\sum_{\sigma\in a} \alpha(\sigma) && \text{using Definitions 2.2.1a and 2.2.2b} \\ &= \textstyle\sum_{\sigma\in S_m(K)} a(\sigma)\alpha(\sigma) && \text{using Definitions 2.2.2a and b.} \end{aligned} \qquad (2.2.$$

Lemma 2.2.4

The Kronecker pairing is bilinear and the map $a \mapsto \langle a, \rangle$ is an isomorphism between $C^m(K)$ and $C_m(K)^\sharp = \hom(C_m(K), \mathbb{Z}_2)$.

Proof

The bilinearity is obvious from the third line of Eq. (2.2.1). Let $0 \neq a \in C^m(K)$. This means that, as a subset of $S_m(K)$, a is not empty. If $\sigma \in a$, then $\langle a, \sigma\rangle \neq 0$, which proves the injectivity of $a \mapsto \langle a, \rangle$. As for its surjectivity, let $h \in \hom(C_m(K), \mathbb{Z}_2)$. Using the inclusion $S_m(K) \hookrightarrow C_m(K)$ given by $\tau \mapsto \{\tau\}$, define

$$a = \{\tau \in S_m(K) \mid h(\tau) = 1\}.$$

For each $\sigma \in S_m(K)$ the equation $h(\sigma) = \langle a, \sigma\rangle$ holds true. As $S_m(K)$ is a basis of $C_m(K)$, this implies that $h = \langle a, \rangle$. ¬

We now define the boundary and coboundary operators. The *boundary operator* $\partial: C_m(K) \to C_{m-1}(K)$ is the $\mathbb{Z}_2$-linear map defined by

$$\partial(\sigma) = \{(m-1)\text{ faces of } \sigma\} = S_{m-1}(\sigma),\ \sigma \in S_m(K). \qquad (2.2.$$

Formula (2.2.2) is written in the language of Definition 2.2.1b. Using Definition 2.2.3, we get

$$\partial(\sigma) = \sum_{\tau\in S_{m-1}(\sigma)} \tau. \qquad (2.2.$$

The *coboundary operator* $\delta: C^m(K) \to C^{m+1}(K)$ is defined by the equation

$$\langle \delta a, \alpha\rangle = \langle a, \partial\alpha\rangle. \qquad (2.2.$$

The last equation indeed defines δ by Lemma 2.2.4 and δ may be seen as the Kronecker adjoint of ∂. In particular, if $\sigma \in S_m(K)$ and $\tau \in S_{m-1}(K)$ then

$$\tau \in \partial(\sigma) \Leftrightarrow \tau \subset \sigma \Leftrightarrow \sigma \in \delta(\tau). \qquad (2.2.$$

The first equivalence determines the operator ∂ since $S_m(K)$ is a basis for $C_m(K)$. The second equivalence determines δ if $S_{m-1}(K)$ is finite. Note that the definition of δ may also be given as follows: if $a \in C^m(K)$, then

$\delta(\alpha) = \{\tau \in S_{m+1}(K) \mid \sharp(\alpha \cap \partial(\tau)) \text{ is odd}\}$
Let $\sigma \in S_m(K)$. Each $\tau \in S_{m-2}(K)$ with $\tau \subset \sigma$ belongs to the boundary of exactly two $(m-1)$-simplexes of σ. Using Eq. (2.2.3), this implies that $\partial\circ\partial = 0$. By Eq. (2.2.4) and Lemma 2.2.4, we get $\delta\circ\delta = 0$. We define the $\mathbb{Z}_2$-vector spaces

- $Z_m(K) = \ker(\partial : C_m(K) \to C_{m-1}(K))$, the m -*cycles* of K.
- $B_m(K) = \text{image}(\partial : C_{m+1}(K) \to C_m(K))$, the m -*boundaries* of K.
- $Z^m(K) = \ker(\delta : C^m(K) \to C^{m-1}(K))$, the m -*cocycles* of K.
- $B^m(K) = \text{image}(\delta : C^{m-1}(K) \to C^m(K))$, the m -*coboundaries* of K.

For example, Fig. 2.1 shows a triangulation K of the plane, with $V(K) = \mathbb{Z} \times \mathbb{Z}$. The bold line is a cocycle α which is a coboundary: $\alpha = \delta B$, with $B = \{\{(m,n)\} \mid (m,n) \in V(K) \text{ and } m \leq 0\}$, drawn in bold dots.

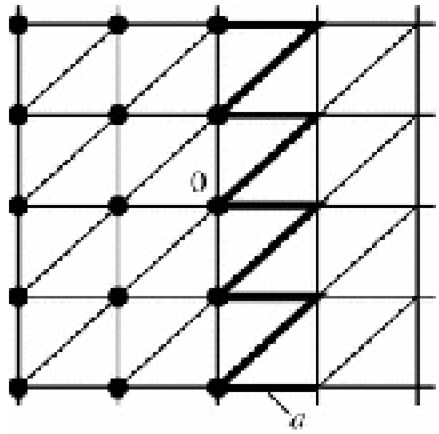

Fig. 2.1 A triangulation K of the plan, with $V(K) = \mathbb{Z} \times \mathbb{Z}$

Since $\partial\circ\partial = 0$ and $\delta\circ\delta = 0$, one has $B_m(K) \subset Z_m(K)$ and $B^m(K) \subset Z^m(K)$. We form the quotient vector spaces

- $H_m(K) = Z_m(K)/B_m(K)$, the mth -*homology vector space of* K.
- $H^m(K) = Z^m(K)/B^m(K)$, the mth -*cohomology vector space of* K.

As for the (co)chains, the notations $H_*(K) = \oplus_{m\in\mathbb{N}} H_m(K)$ and $H^*(K) = \oplus_{m\in\mathbb{N}} H^m(K)$ stand for the (co)homology seen as graded $\mathbb{Z}_2$-vector spaces. By convention, $H_{-1}(K) = H^{-1}(K) = 0$. Also, the homology and the cohomology are in duality via the Kronecker pairing:

Proposition 2.2.5
(Kronecker duality) The Kronecker pairing on (co)chains induces a bilinear map

$H^m(K) \times H_m(K) \xrightarrow{\langle\,,\,\rangle} \mathbb{Z}_2$
Moreover, the correspondence $a \mapsto \langle a, \rangle$ is an isomorphism

$$H^m(K) \xrightarrow[\cong]{\mathbf{k}} \hom(H_m(K), \mathbb{Z}_2) .$$

Proof

Instead of giving a direct proof, which the reader may do as an exercise, we will take advantage of th more general setting of *Kronecker pairs*, developed in the next section. In this way, Proposition 2.2.5 follows from Proposition 2.3.5. ⊔

2.3 Kronecker Pairs

All the vector spaces in this section are over an arbitrary fixed field $\mathbb{F}$. The dual of a vector space V denoted by $V^\sharp$.

A *chain complex* is a pair (C_*, ∂), where

- C_* is a graded vector space $C_* = \bigoplus_{m \in \mathbb{N}} C_m$. We add the convention that $C_{-1} = 0$.
- $\partial : C_* \to C_*$ is a linear map of degree -1, i.e. $\partial(C_m) \subset C_{m-1}$, satisfying $\partial \circ \partial = 0$. The operator ∂ is called the *boundary* of the chain complex.

A *cochain complex* is a pair (C^*, δ), where

- C^* is a graded vector space $C^* = \bigoplus_{m \in \mathbb{N}} C^m$. We add the convention that $C^{-1} = 0$.
- $\delta : C^* \to C^*$ is a linear map of degree $+1$, i.e. $\delta(C^m) \subset C^{m+1}$, satisfying $\delta \circ \delta = 0$. The operator δ is called the *coboundary* of the cochain complex.

A *Kronecker pair* consists of three items:

(a) a chain complex (C_*, ∂).

(b) a cochain complex (C^*, δ).

(c) a bilinear map

$$C^m \times C_m \xrightarrow{\langle\, ,\, \rangle} \mathbb{F}$$

satisfying the equation

$$\langle \delta a, \alpha \rangle = \langle a, \partial \alpha \rangle . \qquad (2.3.$$

for all $a \in C^m$ and $\alpha \in C_{m+1}$ and all $m \in \mathbb{N}$. Moreover, we require that the map $\mathbf{k} : C^m \to C^\sharp_m$, give by $\mathbf{k}(a) = \langle a, \rangle$, is an isomorphism.

Example 2.3.1

Let K be a simplicial complex. Its simplicial (co)chain complexes $(C^*(K), \delta)$, $(C_*(K), \partial)$, together wi