Sergey Kitaev

Patterns in Permutations and Words

Monographs in Theoretical Computer Science
An EATCS Series

For further volumes:
http://www.springer.com/series/776

Sergey Kitaev

Patterns in Permutations and Words

Springer

Dr. Sergey Kitaev
Department of Computer and Information Sciences
University of Strathclyde
Livingstone Tower
26 Richmond Street
Glasgow, G1 1XH
United Kingdom
sergey.kitaev@cis.strath.ac.uk

ISSN 1431-2654
ISBN 978-3-642-17332-5 e-ISBN 978-3-642-17333-2
DOI 10.1007/978-3-642-17333-2
Springer Heidelberg Dordrecht London New York

Library of Congress Control Number: 2011937054

ACM Computing Classification (1998): G.2, F.2

Printed on acid-free paper

Springer is part of Springer Science+Business Media (www.springer.com)

Foreword

The study of patterns in permutations and words has a long history. For example, starting in the 1880s, MacMahon began his study of combinatorial objects. In particular, he gave generating functions for the distribution of inversions in permutations and words which in modern day terminology correspond to occurrences of the pattern 21. The study of descents or consecutive occurrences of the pattern 21 goes back even further in that Leonhard Euler in 1749 introduced polynomials of the form $\sum_{k=0}^{n}(k+1)^n t^k = \frac{A_n(t)}{(1-t)^{n+1}}$ to help in the evaluation of the Dirichlet η-function $\eta(s) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^s}$. The polynomial $A_n(t)$, which is now known as the Eulerian polynomial, is the generating function of the number of descents over the symmetric group S_n, i.e. $A_n(t) = \sum_{\sigma \in S_n} x^{\mathrm{des}(\sigma)}$. The Eulerian polynomials were studied from a combinatorial point of view by Foata and Schützenberger in 1970. The study of permutations and words that have regular patterns also has a long history. For example, André in 1879 showed that the exponential generating function of up-down permutations is $\sec(x) + \tan(x)$ and Carlitz in 1973 found the generating functions for up-down words. These and other works led to the study of permutation statistics which became an active area of research starting in the 1970s and remains an active area of research up to this day.

The origin of the modern day study of patterns in words can be traced back to papers by Rotem, Rogers, and Knuth in the 1970s and early 1980s. The first systematic study of permutation patterns was not undertaken until the paper by Simion and Schmidt which appeared in 1985. Today the study of patterns in permutations and words is a very active field as is evidenced by the exceedingly long bibliography for this volume. At this point, there is a rich collection of tools that has been developed to study a variety of problems such as how to count the number of permutations and words that avoid a given pattern or collection of patterns or how to find the generating function for the number of occurrences of a pattern or collection of patterns. The notion of patterns in permutations and words has proved to be a useful language in a variety of seemingly unrelated problems including the theory of Kazhdan-Lusztig polynomials, singularities of Schubert varieties, Chebyshev polynomials, rook polynomials for Ferrers boards, and various sorting algorithms including sorting stacks and sortable permutations. In addition, the study of patterns in permutations and words also arises in computational biology and theoretical physics.

The present book is a welcome addition to the literature on permutation patterns. In this book, Sergey Kitaev not only systematizes and organizes a vast number of results on patterns in permutations and words that have appeared in the literature, but he also describes the connections of the study of patterns in permutations and words with the various fields mentioned in the previous paragraph. The study of patterns in permutations and words also leads to a large number of interesting questions in bijective combinatorics. For example, we say that two permutations σ and τ are Wilf-equivalent if for all n, the number of permutations of S_n which avoid σ, i.e. which have no occurrences of the pattern σ, is the same as the number of permutations of S_n which avoid τ. There is a vast literature on classifying which pairs of permutations are Wilf-equivalent and such a classification naturally leads to questions of how to find bijective proofs of such equivalences. Similarly, there is a large literature which shows that various other combinatorial objects such as partially ordered sets, lattice paths, planar maps, and labeled trees are equinumerous with certain classes of permutations which avoid a given pattern. This book provides a valuable introduction to the study of bijective questions that arise in the study of patterns in permutations and words. Another novel feature of this book is that Kitaev puts particular emphasis on various generalizations of classical patterns in permutations and words such as partially ordered patterns, barred patterns, vincular patterns, bivincular patterns, and place-difference-value patterns. This is the first book that provides a comprehensive look at such generalizations of classical patterns. Thus, this book provides a welcome and valuable addition to the study of patterns in permutations and words. I believe that it will provide a valuable introduction for graduate students and researchers who want to pursue research in the study of patterns in permutations and words as well as a valuable reference for experts in the field.

Jeffrey B. Remmel
San Diego, February 20, 2011

Preface

This book deals with occurrences of patterns in permutations and, more generally, in words with repeated letters. An occurrence of a pattern in a word is a subsequence of the word whose letters appear in the same order of size as the letters in the pattern. As a simple example, an occurrence of the pattern 321 is simply a decreasing subsequence of length 3, such as the one formed by the letters 632 in the permutation 4631725.

The patterns we deal with have been studied sporadically, often implicitly, for over a century, but in the last 20 years or so, the area has grown dramatically, resulting in many hundreds of published papers and in the annual conference "Permutation Patterns" organized for the first time at the University of Otago in Dunedin, New Zealand, in 2003. The introduction of the area of (permutation) patterns is traditionally attributed to Donald Knuth and in particular to exercises on pages 242–243 in his first volume of "The Art of Computer Programming" [540] in 1968, while the first systematic study of pattern avoidance was done by Rodica Simion and Frank W. Schmidt [721] in 1985.

There are several survey papers on this subject [136, 183, 515, 512, 352, 749, 751, 805] and the entire Volume 9 (2) of the *Electronic Journal of Combinatorics* is devoted to it. Moreover, there are proceedings of the Permutation Patterns 2007 conference, edited by Steve Linton, Nik Ruškuc and Vincent Vatter, and published in the London Mathematical Society Lecture Note Series, Cambridge University Press (vol. 376). The book "A walk through combinatorics" [141] by Miklós Bóna contains material on permutation patterns, while the book "Combinatorics of permutations" [137] by the same author provides a comprehensive and accessible introduction to so-called *classical permutation patterns*. Finally, there is the book "Combinatorics of compositions and words" [461] by Silvia Heubach and Toufik Mansour directly related to the subject.

However, the area has grown far beyond the content of the books mentioned above. The notion of a "pattern" has been extended in many different ways, often bringing new connections to other disciplines. Even such an important and relatively well-studied class of patterns as "generalized patterns" (called "vincular patterns" in this book) introduced by Eric Babson and Einar Steingrímsson in 2000, received almost no attention in the books by Bóna and was not considered for permutations in the book by Heubach and Mansour. One of the goals of this book is to introduce a new notation for vincular patterns, because it has been somewhat confusing so far.

The two main objectives of this book are the following:

1. **To provide a motivation** to study patterns by demonstrating as many links to other areas of research as possible — Chapters 2 and 3 are entirely dedicated to this, and many other parts of the book contain such motivating material. These links provide connections to different combinatorial structures appearing in the literature, but also references to computer science (*sorting*, *generating*, and *complexity issues*), statistical mechanics (*Partially Asymmetric Simple Exclusion Process*), and computational biology (*whole genome duplication-random loss model*).

2. **To be comprehensive** in mentioning existing publications, and in sketching new research directions and trends in the field. In particular, we hope to have gathered, in our bibliography, almost all published papers related to the area (the book contains more than 800 references). Of course, there is a price to pay for being comprehensive, while keeping the size of the book within reasonable limits — we give very few proofs and there are no exercises. However, references are given to all results mentioned, so that the interested reader should have no problems finding the details.

While the book mentions several original results from papers in preparation, a couple of important topics were not covered the way they deserve to be. These topics include, but are not limited to, the intensively studied theory of *pattern classes* (see Remark 6.1.65 for a collection of references on it) and *enumeration schemes* (see [662, 663, 781, 816]). Moreover, we do not discuss at all a couple of research directions, for example, the results on the Möbius function on posets defined by different notions of pattern containment (see [107, 122, 202, 513, 708, 752]). Except for that, the book is a comprehensive collection of up-to-date results on patterns, most of which will be accessible to a broad audience, from undergraduate students to active researchers in the area of patterns in words and permutations, or adjacent fields.

The book is organized as follows.

- In Chapter 1 we introduce the main classes of patterns of interest in this book (classical patterns, barred patterns, vincular patterns, bivincular patterns, and partially ordered patterns) and also provide examples of typical problems on these patterns. Moreover, we list a bibliography related to each of the pattern classes.

- In Chapters 2 and 3 we provide motivation points to study patterns in words and permutations. They include links to theoretical computer science through several sorting devices, planar maps and relevant objects, Schubert varieties and Kazhdan-Lusztig polynomials, computational biology through the tandem duplication-random loss model, statistical mechanics through the Partially Asymmetric Simple Exclusion Process, the theory of partially ordered sets, classification of Mahonian statistics, bijective combinatorics through encoding combinatorial objects by pattern-restricted permutations, and more.

- In Chapter 4 we present the more than thirty year history of bijections between permutations avoiding any classical pattern trivially equivalent to 321 and any pattern trivially equivalent to 132. We discuss a recent classification of these bijections and a philosophical question on what is a "good" bijection from the point of view of bijective combinatorics. Additionally, this chapter provides a collection of approaches to deal with classical patterns, along with general theorems about Wilf-equivalence of certain classical patterns.

- Chapter 5 contains an overview of results on consecutive patterns, and various approaches to studying them. For example, in this chapter we discuss the symbolic method, the symmetric functions approach, and the cluster and chain methods.

- In Chapters 6 and 7 we collect known (mostly enumerative) results on various patterns other than consecutive ones.

- In Chapter 8 we discuss several topics without a common thread. These topics include simple permutations, pattern matching problems, Gray codes, packing patterns, a link to combinatorics on words, universal cycles, simsun permutations, and games on patterns in permutations.

- In Chapter 9 we present several extensions and generalizations of the study of patterns discussed in the previous chapters.

- Appendix A serves as an easy access to the definitions for most of the permutation/word statistics and number sequences appearing in this book, while in Appendix B we provide a couple of basic algebraic facts, including the notion of Chebyshev polynomials of the second kind.

Acknowledgments

A significant part of the book was written during the author's six month visit to the Polytechnic University of Catalonia, Barcelona, Spain, in Summer-Fall 2010. The visit was supported by a grant from Iceland, Liechtenstein and Norway through the EEA Financial Mechanism, and supported and coordinated by Universidad Complutense de Madrid. I am grateful to Marc Noy and Anna de Mier for their hospitality during my stay in Barcelona.

I am deeply indebted to the following mathematicians who kindly agreed to carefully proofread different parts of the book, providing substantial feedback: Sara Billey, Alexander Burstein, Mark Dukes, Jeff Liese, Aaron Robertson, Christopher Severs, Rebecca Smith, Einar Steingrímsson, Vince Vatter and Lauren Williams.

Special thanks go to Lara Pudwell and Manda Riehl, who each proofread two chapters providing valuable comments, to Phil Watson who went through the entire book giving many useful suggestions, and to Henning Úlfarsson, who not only proofread a section of this book, but also provided technical assistance related to LaTeX issues. Moreover, I would like to thank the following individuals for helpful discussions related to the book: Mike Atkinson, Anders Claesson, Vít Jelínek, Eva Jelínková, Marc Noy and Don Rawlings.

I would like to thank my parents, Polina Kitaeva and Alexander Filin, my grandmother Fedocia Semenova (who is about to turn 93!) and my parents-in-law, Sergey Shuiskii and Nina Shuiskaya, for their constant support. I also want to thank all my friends and colleagues for making my life so interesting.

The final acknowledgements go to my wonderful wife Daria Kitaeva and my son Daniel Kitaev for their love and inspiration. It is to them I dedicate this book.

Sergey Kitaev
Reykjavík, March 11, 2011

Notation

$\lvert A \rvert$	the number of elements in a set A
$\mathrm{asc}(\pi)$	the number of ascents in π
$\mathrm{Av}(P)$	the set of all permutations avoiding each pattern in P
$\mathrm{Av}_\ell(P)$	the set $\cup_{n\geq 0} W_{n,\ell}(P)$
b.g.f.	bivariate generating function
BP	barred pattern
BVP	bivincular pattern
C_n	the n-th Catalan number
$C(x)$	the g.f. $\frac{1-\sqrt{1-4x}}{2x}$ for the Catalan numbers
$c(\pi)$	the complement of a permutation π
$\mathcal{D}_n$	the set of all Dyck paths on $2n$ steps
$\mathrm{des}(\pi)$	the number of descents in π
$\mathbb{E}$	the set of even numbers $\{0, 2, 4, \ldots\}$
e.g.f.	exponential generating function
F_n	the n-th Fibonacci number
$f.g(\pi)$	the composition $f(g(\pi))$
g.f.	generating function
$i(\pi)$	the inverse of a permutation π
$\mathrm{inv}(\pi)$	the number of inversions in a permutation π
$[\ell]$	the set $\{1, 2, \ldots, \ell\}$
$[\ell]^n$	the set of all words of length n over $[\ell]$
$[\ell]^*$	the set $\cup_{n\geq 0}[\ell]^n$
M_n	the n-th Motzkin number
$\mathbb{N}$	the set of natural numbers $\{0, 1, 2, \ldots\}$
n-permutation	permutation of length n
$[n]_q$	$1 + q + q^2 + \cdots + q^{n-1} = \frac{1-q^n}{1-q}$
$[n]_{p,q}$	$\frac{p^n-q^n}{p-q}$
$[n]_q! = [n]!$	$[n]_q[n-1]_q \cdots [1]_q$
$[n]_{p,q}!$	$[n]_{p,q}[n-1]_{p,q} \cdots [1]_{p,q}$
$\begin{bmatrix} n \\ k \end{bmatrix}_q$	$\frac{[n]_q!}{[k]_q![n-k]_q!}$

$\mathbb{O}$	the set of odd numbers $\{1, 3, 5, \dots\}$
$\mathbb{P}$	the set of positive integers
POP	partially ordered pattern
$\mathbb{Q}$	the set of rational numbers
$\mathbb{R}$	the set of real numbers
$r(\pi)$	the reverse of a permutation/word π
$\mathrm{red}(\pi)$	the reduced form of a permutation π
$\mathcal{S}_n$	the set of all permutations of length n
S_n	the n-th (large) Schröder number
$\mathcal{S}_n(P)$	the set of all n-permutations avoiding each pattern in P
$\mathcal{S}_n^k(P)$	the set of all n-permutations containing k occurrences of patterns in P
$\mathcal{S}_\infty$	the set $\cup_{n\geq 0}\mathcal{S}_n$
$s_n(P)$	$\lvert\mathcal{S}_n(P)\rvert$
$s_n^k(P)$	$\lvert\mathcal{S}_n^k(P)\rvert$
$S(n,k)$	Stirling number of the second kind
$U_n(x)$	the n-th Chebyshev polynomial of the second kind
VP	vincular pattern
$W_{n,\ell}(P)$	the set of all words in $[\ell]^n$ that avoid each pattern in P
$W_{n,\ell}^k(P)$	the set of all words in $[\ell]^n$ that contain exactly k occurrences of patterns in P
$w_{n,\ell}(P)$	$\lvert W_{n,\ell}(P)\rvert$
$w_{n,\ell}^k(P)$	$\lvert W_{n,\ell}^k(P)\rvert$
$\mathbb{Z}$	the set of integers
$\alpha_1 \oplus \alpha_2$	$12[\alpha_1, \alpha_2]$
$\alpha_1 \ominus \alpha_2$	$21[\alpha_1, \alpha_2]$
ε	the empty word/permutation
$p_1 \sim p_2$	p_1 and p_2 are Wilf-equivalent
$p_1 \sim_k p_2$	p_1 and p_2 are k-Wilf-equivalent
$p_1 \sim_* p_2$	p_1 and p_2 are strongly Wilf-equivalent
$\sigma[\alpha_1, \alpha_2, \dots, \alpha_m]$	the inflation of σ by $\alpha_1, \alpha_2, \dots, \alpha_m$

Contents

Chapter 1

What is a pattern in a permutation or a word?

We begin with some basic definitions.

Definition 1.0.1. A *word* is a sequence whose symbols (or *letters*) come from a set called an *alphabet*. Alphabets in this book are finite, and the most typical alphabet here is of the form $[\ell] = \{1, 2, \ldots, \ell\}$. Most of the words we deal with in this book are finite as well. We let $[\ell]^n$ denote the set of all words of length n over $[\ell]$ and $[\ell]^* = \cup_{n\geq 0}[\ell]^n$. A word from an ℓ-letter alphabet is often referred to as an "*ℓ-ary word*".

Example 1.0.2. 2411121 and 25554 are words over the alphabet $[5] = \{1, 2, 3, 4, 5\}$, while *abbacca* is a word over the alphabet $\{a, b, c\}$.

$$[3]^2 = \{11, 12, 13, 21, 22, 23, 31, 32, 33\}.$$

It is easy to show that there are k^n different words of length n over a k-letter alphabet.

Definition 1.0.3. A *permutation* of length n (also referred to as an n-permutation) is a one-to-one function from an n-element set to itself. We write permutations as words $\sigma = \sigma_1\sigma_2\cdots\sigma_n$, whose letters are distinct and usually consist of the integers $1, 2, \ldots, n$ (where σ_i is the image of i under σ). This way of writing permutations is often referred to as *one-line notation*. We let $\mathcal{S}_n$ denote the set of all permutations of length n, and $\mathcal{S}_\infty = \cup_{n\geq 0}\mathcal{S}_n$.

Example 1.0.4. 413265 is a 6-permutation, whereas 265, *bca*, and 132 are permutations of length 3. $\mathcal{S}_1 = \{1\}$, $\mathcal{S}_2 = \{12, 21\}$, $\mathcal{S}_3 = \{123, 132, 213, 231, 312, 321\}$, etc.

S. Kitaev, *Patterns in Permutations and Words*, Monographs in Theoretical Computer Science. An EATCS Series, DOI 10.1007/978-3-642-17333-2_1,