Taking Sudoku Seriously

The Math Behind the World's Most Popular Pencil Puzzle

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In memory of Martin Gardner, who showed a generation of mathematicians the value of puzzles as a gateway into mathematics.

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PREFACE

Every math teacher knows the frustration of directing a seemingly simple question to a class and receiving blank stares in return. In part, this reaction can be attributed to general student apathy or to fear of giving the wrong answer. There is, however, a more fundamental issue to be addressed.

Most people, when asked to describe mathematics, will talk about the tedious algorithms of arithmetic or the seemingly arbitrary rules of algebra. For them it is all about symbol manipulation and mindless computation. This view is entirely understandable given that they probably saw little else in their grade school and high school mathematics classes.

Mathematicians do not recognize their discipline in such descriptions. We see arithmetic and algebra as tools used in doing mathematics, just as hammers and handsaws are tools used in carpentry. For professionals, mathematics is about curiosity, imagination, and solving problems. There are questions that are instinctive and natural for mathematicians that rarely occur to those looking in from the outside. There is such a thing as a mathematical view of the world. Sadly, it is a view that is too often hidden from those struggling to learn the subject.

Which brings us back to the blank stares. Often the problem is simply that mathematicians have a way of expressing themselves that makes little sense to those outside the club. Students unaccustomed to the sorts of questions mathematicians ask, or unaware that mathematics is about asking questions in the first place, will often be confused by questions more experienced people regard as simple. We need first to develop mathematical thinking in our students before we expect them to toss off answers to our questions.

That is where the Sudoku puzzles come in. We can define a *Sudoku square* as a 9×9 grid in which every row, column, and 3×3 block contains the digits 1–9 exactly once. A *Sudoku puzzle* is then a square in which some of the cells have been filled in while others are blank. The goal of the solver is to fill in the blank cells in such a way that the result is a Sudoku square. If the puzzle is sound there will only be one way of doing that.

Here is an example. This is a level 3 puzzle, where level 1 is the easiest and level 5 is the hardest.

Puzzle 1: Sudoku Warm-Up.

Fill in the grid so that each row, column, and block contains each of the numbers 1–9 exactly once. The solution to this puzzle is at the end of the book.

	2	3						
	5		7				8	3
		4				2		7
				7			6	
			9		6			
	3			2				
3		5				8		
6	1				8		7	
						4	1	

Over the past five years, Sudoku puzzles have become a mainstay of many newspapers. Such venues are typically careful to assure the reader that, the presence of numbers notwithstanding, Sudoku puzzles are not math problems. They are keen to stress that any collection of nine distinct symbols, such as the first nine letters of the alphabet, would work just as well.

This sort of thing sounds bizarre to a mathematician. In saying that Sudoku does not involve mathematics, the newspaper really means it does not involve arithmetic. The sort of reasoning that goes in to solving a Sudoku puzzle, on the other hand, is at the heart of what mathematics is all abou That so many people will claim to hate doing mathematics while simultaneously enjoying the challenge of solving a puzzle is a source of frustration to those of us in the business.

To a mathematician, Sudoku puzzles immediately suggest a whole host of interesting questions even beyond the reasoning that goes in to solving them. How many Sudoku squares are there? What sorts of transformations can you do to a Sudoku square to produce other such squares? What is the smallest number of initial clues a sound puzzle can have? What is the largest number of initial clues puzzle can have without having a unique solution? Is it possible to have a Sudoku square in which each 3 × 3 block is actually a semimagic square (so that the digits in each row and column within the block add up to the same sum)? In attempting to answer these questions we will inevitably encounte a lot of interesting mathematics.

More than that, however, we will use Sudoku puzzles and their variants as a gateway into mathematical thinking generally. This is both a math book and a puzzle book. The puzzles, in additio to being enjoyable simply as stand-alone brainteasers, will serve to complement and introduce the mathematical concepts in the text. Our emphasis throughout is on asking questions and solving problems; technical mathematical machinery will be introduced only as it arises naturally in the course of our reasoning.

We have a number of different audiences in mind. For students in high school or college we inten to provide a view of mathematics that is very different from what is usually presented. It is a far mo realistic view than the one implied by years of training in tedious symbol manipulation. For educato we hope to provide some novel ideas for how to bring genuine mathematical thinking into the classroom in a context that will be interesting and accessible to students. For any layperson with a general interest in mathematics, we provide plenty of food for thought and intellectual stimulation. Professional mathematicians can benefit from seeing familiar mathematical abstractions applied in novel settings.

We have assumed little beyond high school mathematics. Indeed, if you flip through the book righ now ~~you will notice that for the most part we make limited use of mathematical symbols. Our focus~~ i on ideas and reasoning; “notions, not notations,” as the saying goes. That is not to say, however, that the book is always easy going. Mathematics takes some getting used to, and you should not be surprised if you have to pause periodically to mull over something we have said. Furthermore, things do get gradually more complex as we go along, and readers without previous mathematical experience might find some of the concluding material a bit more challenging than what came befor Even here, though, we believe we have provided enough commentary to make the central ideas comprehensible to all. In those few cases where we have elected to include some more technical material, the dense calculations can be skimmed over without losing the thread of the discussion.

The book is structured as follows: In the first chapter we examine techniques for solving Sudoku puzzles and discuss the general question of what constitutes a math problem. Chapter 2 discusses the notion of a Latin square, an object of long-standing interest to mathematicians of which Sudoku squares are a special case. Chapter 3 discusses Greco-Latin squares, which are an extension of the idea of a Latin square. Chapters 4 and 5 discuss two counting problems related to Sudoku. Specifically, we determine the total number of Sudoku squares and the total number of “fundamentally different” squares. In the course of this discussion, we cannot avoid presenting fundamental ideas from combinatorics and abstract algebra. Chapter 6 presents the problem of how one finds interesting Sudoku puzzles and places this problem within the context of search problems generally. Chapters 7 and 8 investigate connections between Sudoku, graph theory, and polynomials. Chapter 9 is an exploration of Sudoku extremes. We look for puzzles with the maximal number of vacant regions, with the minimal number of starting clues, and numerous others. The book concludes with a gallery of novel Sudoku variations. No math here, just pure solving fun! All of the puzzles presented in the text, save for a handful of exceptions that are explicitly identified, are original to thi volume.

A final, bureaucratic detail. The solutions to many of the puzzles appear in the back of the book. I some cases, however, the solution to the puzzle is essential to the exposition and, therefore, has been included in the text. Wherever possible, we have placed the solution to a puzzle on a different page from the puzzle itself. Occasionally this was not possible. For that reason you may find it useful to read with an index card in hand. This will allow you to conceal portions of the page you do not wish to read immediately.

The history of math and science shows there is often great insight to be gained from the earnest consideration of trivial pursuits. Probability theory is today an indispensable tool in many branches o science, but it was born out of gambling and games of chance. In the early days of computer science and artificial intelligence, much attention was given to the relatively unimportant problem of programming a computer to play chess.

We have similar ambitions for this book. Perhaps you have tended to see Sudoku puzzles as an amusing diversion, useful only for passing the time during long airplane rides. After reading this boo you will see instead a gateway into the world of mathematics. It is a far different, and more beautiful world than you may think.

The authors would like to thank Philip Riley, whose computer prowess assisted greatly with the construction of many of the Sudoku puzzles in this book. Without Phil’s work at Brainfreeze Puzzles large portions of this book would not exist. We would also like to thank our Sudoku Master beta-tester Rebecca Field for checking all of the puzzles in the text for accuracy and playability. Finally, we would like to thank Phyllis Cohen, our editor at Oxford University Press, who was tremendously helpful and supportive throughout the writing of this book.

1

Playing the Game
Mathematics as Applied Puzzle-Solving

What is it about puzzles that makes them so engrossing?

Imagine you are minding your own business, thinking your very practical and familiar thoughts, when someone challenges you to measure an interval of nine minutes using only a four-minute hourglass and a seven-minute hourglass. You are dismissive, perhaps, protesting you have little time for such frivolity. But the question gnaws at you, and pretty soon you are wondering what happens if you start both hourglasses going at the same time. You notice that when the four-minute glass runs out, there are three minutes left in the seven-minute glass, and then you start looking for ways to turn that to your advantage. I can time three, four, and seven minutes, you think, but how does that help me get nine? Then you are gone, your formerly practical thoughts banished until the problem is solved.

Or maybe you are presented with two bottles, one containing a liter of water, the other a liter of wine. You are told that some amount of wine is transferred to the water bottle and the resulting water wine mixture is thoroughly stirred. Enough of this mixture is now transferred back to the wine bottle so that both bottles again possess one liter of liquid. Is there now more water in the wine bottle or wine in the water bottle? That the question seems unanswerable is part of its charm. Seriously, what can we do? Having no idea how much liquid was transferred, it would seem I can not determine eithe of the quantities in question. But there must be *something* I can do, as it would be a serious breach of etiquette to present a puzzle with no solution. Maybe there is something in the fact that it was pure wine that was transferred to the water bottle, as opposed to a dilute water-wine mixture that was transferred back … and once having started down this path, you would do well to cancel your remaining appointments for the day. (Solutions to both of these puzzles are presented in Section 1.6.

Or maybe you are shown a 9 × 9 grid like this one:

7						2	9	
				3	2			6
	1					5		3
	5		1			*C*	2	
	7	4	*B*	6	*A*	8	5	
	2				3		7	
6		7					3	
1			6	9				
	9	5						2

You are challenged to fill in the vacant cells with the digits 1–9 in such a way that each row, column, and 3 × 3 block contains each digit exactly once.

That this is surely the most trivial of pursuits does not stop you from noticing that cell *A* has rathe a lot of digits surrounding it. Certainly *A* can not be a 2 or 3, since those digits already appear in its column. Its row brings the digits 4–8 to the party, while its 3 × 3 block puts the kibosh on 1. This leaves 9 as the only possibility, and we happily pencil it in.

Perhaps now you notice the 2s in rows 4 and 6. They are shooting out horizontal laser beams that will burn your fingers if you try to place a 2 in the fourth or sixth rows of the central 3 × 3 block. Bu there must be a 2 *somewhere* in that block, and with the 9 filled in that leaves only cell *B*.

~~Suddenly all of the occupied cells are shooting out lasers! The 3s in row 6 and column 9 have~~ the center-left 3 × 3 block so sliced up that the only place for its 3 is cell *C*. This is turning out to be so much fun that we had better put all else on hold until the remaining cells yield forth their secrets.

This, as you are probably aware, is an example of a Sudoku puzzle. In recent years, they have become enormously popular. Newspapers routinely present them right alongside the venerable crossword puzzle, and in-flight magazines are seldom found without them. The puzzle sections of bookstores are dominated by anthologies of Sudoku puzzles. There are countless websites devoted to Sudoku and its variants, and there are public competitions where people race to solve them.

And if there is one thing about which all of these venues agree it is that, the necessity for writing actual numbers in those little cells notwithstanding, solving a Sudoku puzzle has nothing to with mathematics. Writing in *Scientific American*, computer scientist Jean-Paul Delahaye [17] provides a blunt statement of the basic view:

> *Ironically, despite being a game of numbers, Sudoku demands not an iota of mathematics of its solvers. In fact, no operation – including addition or multiplication – helps in completing a grid, which in theory could be filled with any set of nine different symbols (letters, colors, icons and so on).*

An interesting argument, and doubtless compelling to those who regard *arithmetic* as synonymous with *mathematics*. Let us suggest, however, that there is quite a leap in going from “no addition or multiplication,” to “not an iota of mathematics.” And if you found anything remotely amusing in our previous discussion, then you have more of a taste for mathematics than you might realize.

1.1 MATHEMATICS AND PUZZLES

Mathematicians are professional puzzle-solvers. We are not professional arithmeticdoers. Our job is to seek out puzzles that amuse us and solve them. People pay us to do this because history shows that the earnest contemplation of amusing puzzles routinely leads to constructs of enormous practical value.

For example, in the seventeenth century, the nobleman and gambler known by his title Chevalier de Méré introduced the “Problem of Points.” Imagine that two people, Alice and Bill, are taking turns flipping a coin. Alice gets a point for each heads, while Bill gets a point for each tails. The winner is the first to ten points, and the score is currently eight to seven in Alice’s favor. Further assume the prize is a pot of money to which Alice and Bill have contributed equally. If the game were suspended at this point, how ought we divide the pot between Alice and Bill?

This problem came to the attention of Blaise Pascal and Pierre de Fermat, two gentlemen who rather enjoyed such puzzles. Fermat observed that since the game will end in no more than four tosses, there were only sixteen ways things can play out. We can simply list them all. We would then find that in eleven out of the sixteen cases, Alice wins, as compared to only five for Bill. Since each of these sixteen scenarios is equally likely, we should give $\frac{11}{16}$ of the pot to Alice and the rest to Bill. Pascal agreed with this division, but then one-upped Fermat by deriving general formulas for each player’s chances in more general scenarios. In so doing they began a line of investigation that led to the modern theory of probability. (See the book by Rosenhouse [34] for more information and further references.)

Then there are the famous paradoxes introduced by the philosopher Zeno in the fifth century BC. One of his puzzles proposed that motion was impossible. You see, in traveling from point *A* to point

B, you must first traverse half the distance. Doing so requires first traversing half of *that* distance or one~~-quarter of the total distance. No matter how small the distance, you must first traverse half~~ of it before completing the trip. It would seem you must carry out infinitely many steps before getting anywhere, and that is why motion is impossible.

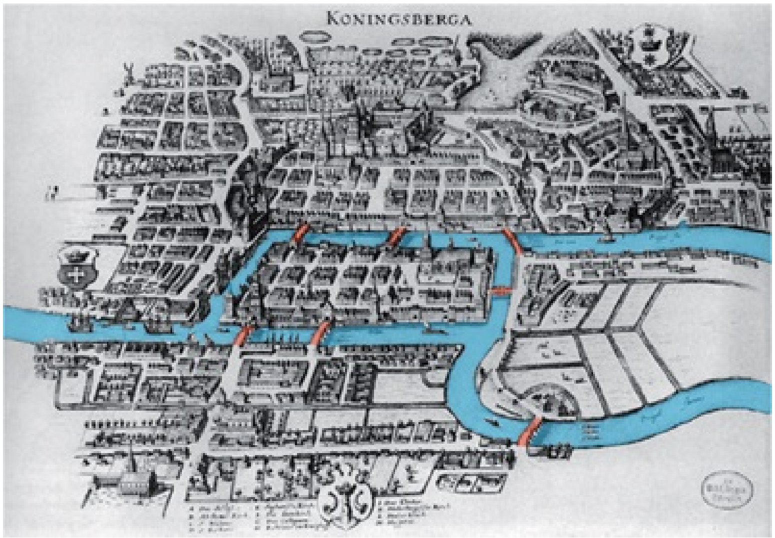
A fully satisfactory response to Zeno was not forthcoming until the seventeenth century, when Isaac Newton and Gottfried Leibniz got to wondering why, exactly, a finite distance could not be divided into infinitely many pieces. Considering a variation on Zeno's paradox, they noted that in traveling a distance of one mile you must first travel half of a mile. Then you must travel half of the remaining distance, or one-quarter of a mile. Then you must travel one-eighth of a mile and so on. Your total distance, which is one mile in this case, is then the sum of these infinitely many smaller steps. That is

$$\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\cdots=1$$

But does this equation make sense? Is there a way of thinking about addition that makes plausible the notion of an infinite sum? Persist in this line of thought and you are well on your way to inventing calculus [15].

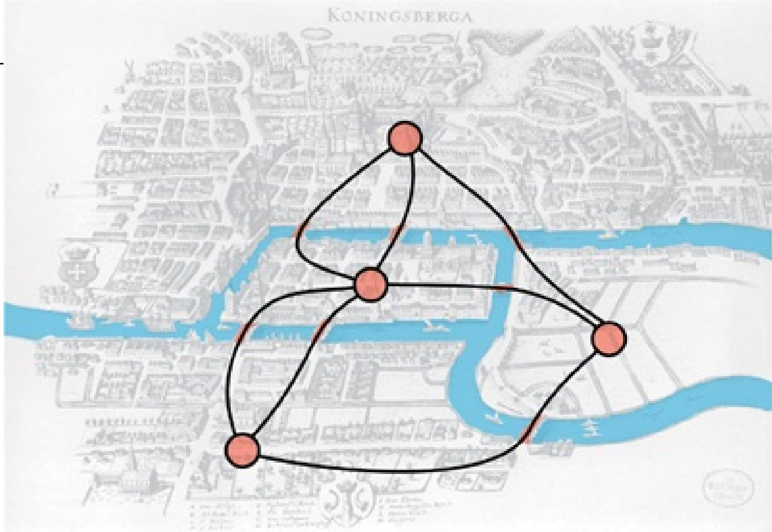
Then there is the famous story of the bridges of Königsburg. It seems that the city of Königsburg in Prussia (known today as Kaliningrad in Russia) was divided into four pieces by the Pregel River. These pieces were linked by seven bridges, as shown in this map:

The Seven Bridges of Königsburg



The locals had gotten to wondering whether it was possible to walk through the city in such a way that each bridge is crossed exactly once. In 1735, Leonhard Euler had the idea of representing the situation via the following abstract model: Each land mass could be thought of as a single location representable as a dot, or *vertex*. Each bridge could be thought of as a line, or *edge*, connecting two o the land mass vertices. The resulting diagram is referred to as a *graph*.

A Graph Representing the Seven Bridges of Königsburg



Euler noticed that every vertex has an odd number of edges coming out of it. Imagine now that yo are walking through the town. Each time you first enter, and then leave, a vertex you "use up" two available edges. That means if there were a complete walk through the town, then it is only the starting and ending vertices that could have an odd number of edges. Since that is not the case here, we see that the locals will search in vain for the desired walk.

The number of edges attached to a vertex is called the *degree* of the vertex. Euler had discovered that in order for there to be a path that travels over every edge exactly once, the graph must have either exactly two vertices with odd degree (the start and end of the path), or no vertices with odd degree (in which case the path starts and ends at the same place). Such walks are now called *Euleriar paths* if they have different starting and ending points, and *Eulerian circuits* if they loop back to thei starting points.

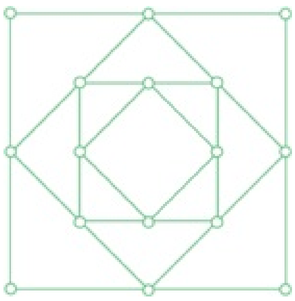
More surprisingly, it turns out that having two or zero vertices of even degree is not just a necessary condition, it is a sufficient condition as well! In other words, if a graph has either exactly two or zero odd-degree vertices, then an Eulerian path or circuit must exist.

Euler's breakthrough was one of those exceedingly clever insights that transforms a puzzle from opaque to crystal clear. It also inaugurated the branch of mathematics known as Graph Theory, whicl remains a going concern to this day.

Here are a couple of puzzles to whet your appetite. Note that Euler's observations can tell you if the graphs below have Eulerian circuits, but they do not tell you how to *find* these circuits. That part is up to you:

Puzzle 2: Eulerian Circuits.

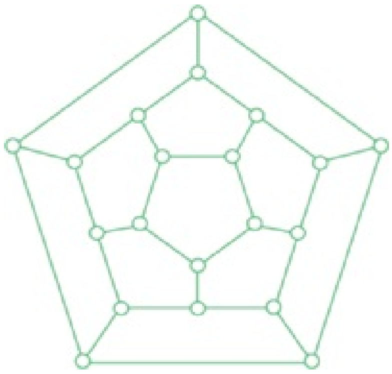
An *Eulerian circuit* for a graph is a path that starts at one vertex, travels along the edges so that it visits every edge exactly once, and then returns to the original vertex. Find one in this graph:



In an Eulerian circuit we traverse each edge exactly once. What if instead we want to reach every *vertex* exactly once? That is, we declare that while we are walking around the town we cannot cross a bridge multiple times, but we drop the requirement of having to traverse every edge of the graph at least once. This might come up if, for example, you were driving to a number of errands at various stores (the vertices) and did not want to retrace your steps on any of the streets (the edges).

Puzzle 3: Hamiltonian Circuits.

A *Hamiltonian circuit* on a graph is a path that starts at one vertex, travels along the edges of the graph so that it visits every *vertex* exactly once, and then returns to where it started. Individual edges can be left untraveled. Find a Hamiltonian circuit in this graph:



Perhaps, as anthropologist Marcel Danesi suggests, humans have an instinct for puzzles just as we have an instinct for language [16]. Maybe it is some side product of our evolution, as though natural selection says, “You can have a big brain, but only at the cost of becoming hopelessly distracted by every silly teaser to come down the road.” Perhaps it is the very frivolity of puzzles that makes them so much fun. After all, were these problems important, our inability to solve them would be cause fo concern, and not bemused frustration.

Whatever the reason, we will take it as given that there is something deeply satisfying in encountering opacity and, using nothing more formidable than your own intellect, producing clarity. With that in mind, let us return to our Sudoku puzzles. We will devote this and the next two sections to considering some techniques for solving them.

1.2 FORCED CELLS

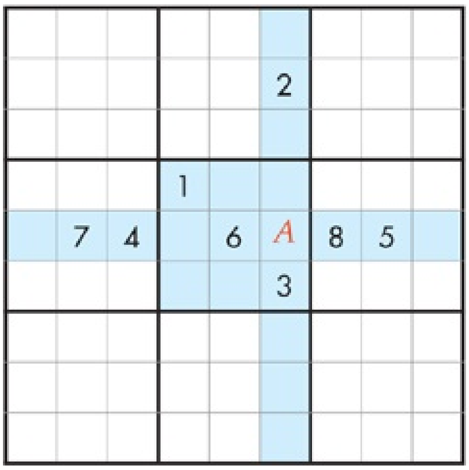
We now revisit our original puzzle:

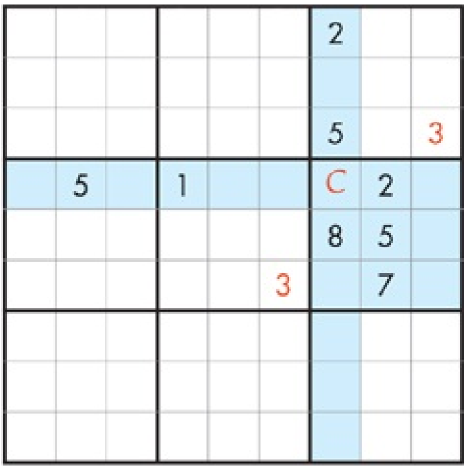
Puzzle 4: Sudoku Walkthrough.

Fill in each grid so that every row, column, and block contains each of the numbers 1–9 exactly once. We will walk through the first half of the solution in the text below.

7						2	9	
				3	2			6
	1					5		3
	5		1				2	
	7	4		6		8	5	
	2				3		7	
6		7					3	
1			6	9				
	9	5						2

By now we have developed a habit of thinking in which we do not see just eighty-one individual cells. Instead, each cell has associated to it a particular zone of the board consisting of its row, column, and 3 × 3 block. For example, for the cells *A* and *C*, we have the following zones:





We shall refer to the cells *A* and *C* as the *generators* of their shaded zones. Our first solving technique used the fact that the digit found in the generating cell must be different from every other digit in the zone. We determined the value of *A* by inspecting its zone and noting that 9 was the only absent digit.

We applied a different technique to *C*. Merely inspecting its zone was inadequate in this case sinc 3, 4, 6, and 9 are all currently missing. Thus, instead of choosing a cell and asking which value it could contain, we selected a particular region (*C*'s 3 × 3 block), noted that it must contain a 3, and asked which of its vacant cells was available for that purpose. The red 3s shown outside of *C*'s zone force *C* to be 3.

These examples suggest a general starting point for solving Sudoku puzzles. Examine the zone of each vacant cell and pencil in all of its candidate values. Having done so, look for the following type of *forced cells*:

1. *One-choice*: A cell that contains only one candidate value.

2. *One-place*: A region (row, column, or block) that has only one cell available for a given number

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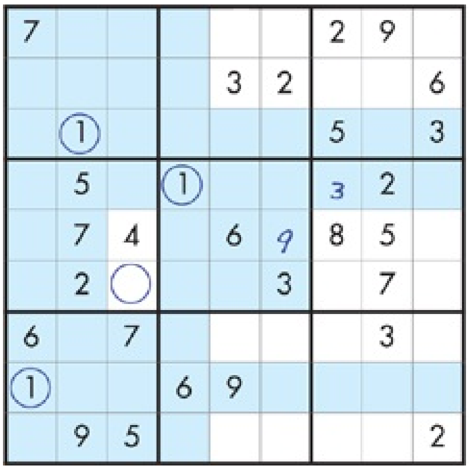
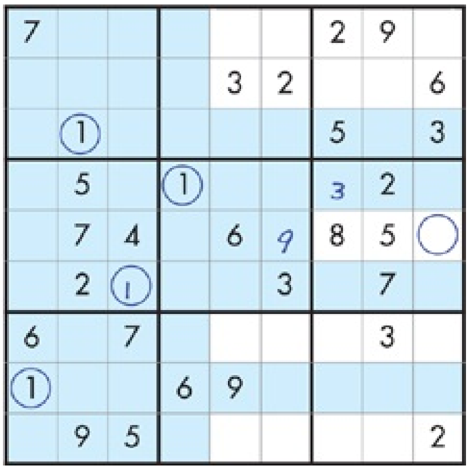
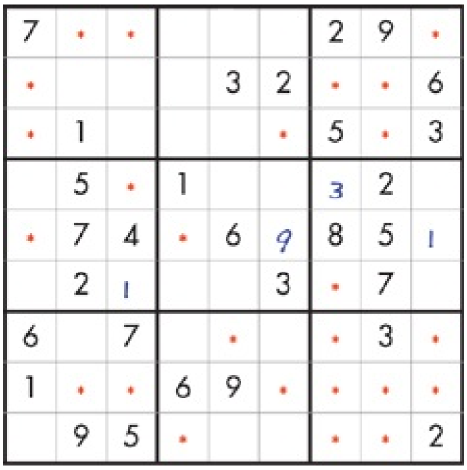
Many easy puzzles can be solved with nothing more than forced cell techniques. Let us try applying them to our current puzzle.

Below left we see the zones for all the 1s that are currently placed. Notice that in the leftmost middle block there is only one place where a 1 could go (indicated by the vacant circle). By the method of one-place we can enter a 1 in that cell. This permits some further shading, which produces another block – the rightmost middle one – in which there is now only one place for a 1.





With nothing more than repeated scanning for one-place situations and filling in any obvious one-choice cells, you can fill in all of the cells with asterisks below. Have a go at it and join us on the other side:



What now? We could keep going using just one-place and one-choice, but let’s look at a new technique.

1.3 TWINS

Our Sudoku board now looks like this:

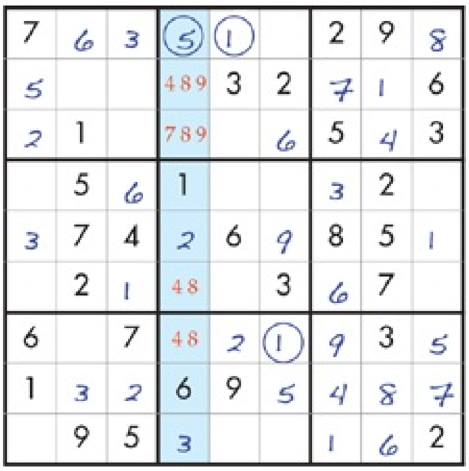
The tiny red values in the seventh row are the candidates for these cells. That is, they are the numbers that are not immediately eliminated by considering the zones of the other cells on the board

Having run out of forced cells, we will have to try something a bit more clever. Instead of looking for cells with only one candidate value, perhaps we should go looking for cells with two. For exampl the first two open cells in the seventh row cannot possibly contain any values other than 4 or 8.

These two cells are an example of a *twin*. They are a pair of cells in the same region having the same two candidate values. Twins are potentially very helpful. In the present case, for example, we can be absolutely certain that between these two cells, one of them contains a 4 while the other contains an 8. If the first one is 4 then the second must be 8, and vice versa.

Look now at the third open cell in the seventh row. Examining zones shows that its only candidate are 1, 4, and 8. But 4 and 8 must already appear in the first two cells of the seventh row, in a currentl unknown order. This tells us that the third open cell in the seventh row must in fact contain a 1. Remarkable! Even without specifying the order of the 4 and 8 in the first two cells, we can determine the value of the third.

Filling in the 1 we just identified, plus the two other circled cells we get by scanning anew for one place situations, we get the board below left. Repeating the method of twins now helps us in the fourth column. Two of its open cells contain only 4 and 8 as candidates. This pair of twins tells us th no other cell in the fourth column can contain a 4 or an 8. This forces the topmost open cell in that column to be 9, which in turn forces the second open cell in that column to be a 7, as shown below right:



The rest of the puzzle should now fall into place rather quickly. Finish it up and join us in the nex section. If you get stuck or want to check your answer, the solution is in the back of the book.

1.4 *X*-WINGS

The previous puzzle was solvable using only forced cells and twins. Sadly, sometimes more is required. Consider, for example, the following puzzle:

Puzzle 5: Harder Sudoku.

Fill in each cell so that every row, column, and block contains each of the numbers 1–9 exactly once. We will walk through the trickier parts of the solution in the text below.

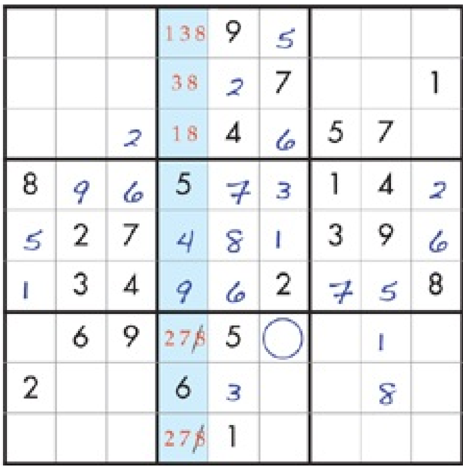
				9				
					7			1
				4		5	7	
8			5			1	4	
	2	7				3	9	
	3	4			2			8
	6	9		5				
2			6					
				1				

The cells with asterisks as shown below left are not so hard to fill in, so let's start with those. The result is shown below right. Cover up the right side and see if you can get there yourself!

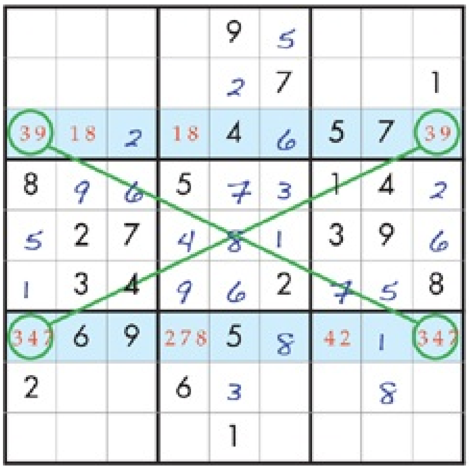
				9	*			
				*	7			1
		*		4	*	5	7	
8	*	*	5	*	*	1	4	*
*	2	7	*	*	*	3	9	*
*	3	4	*	*	2	*	*	8
	6	9		5			*	
2			6	*			*	
				1				

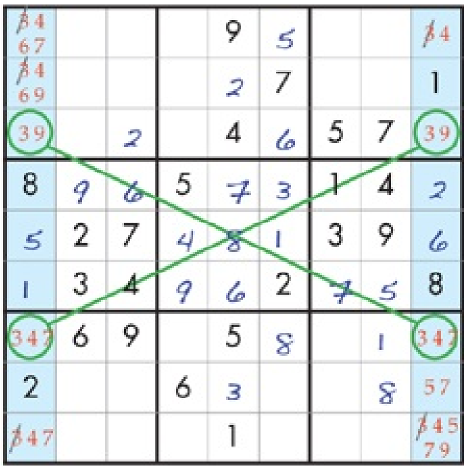
				9	5			
				2	7			1
		2		4	6	5	7	
8	9	6	5	7	3	1	4	2
5	2	7	4	8	1	3	9	6
1	3	4	9	6	2	7	5	8
	6	9		5			1	
2			6	3			8	
				1				

Things get tricky now, but here is one way to proceed. As shown below left, a set of twins in the third row allows us to eliminate a 3 from consideration in the fourth cell of that row. Now look below right. What worked for pairs of numbers works just as well for triples. The first three cells of the fourth column have candidate values entirely among 1, 3, and 8, implying that no other cell in that column can contain those values. This allows us to remove the candidate 8s from the last two open cells in the fourth column. If we now look at the circled cell, we see that it is the only one in its row that can contain an 8.



This brings us to the strategy known as *X-Wings*. Look at the possible candidates in the third and seventh rows shown below left. In each of these rows the 3 can only appear in the first or last cell. That means the 3 in the first column appears either in row 3 or row 7, and likewise in the last column Therefore, as shown below right, we can eliminate all candidate 3s from the remaining cells in those columns. This allows us to enter a 4 in the northeast corner of the puzzle!



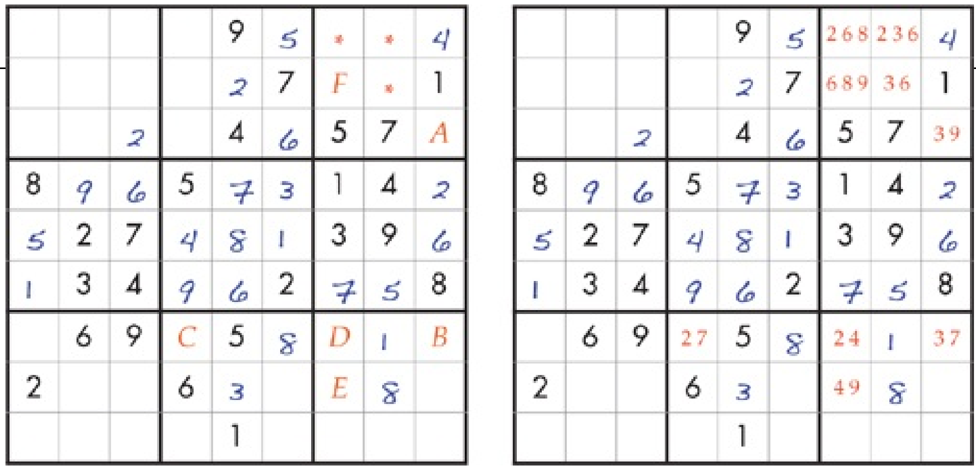


Notice that the four cells with green circles form the corners of a rectangle. Given a 3 in one corner, the diagonally opposite corner is forced to contain a 3 as well. Since the relationships betwee pairs of opposite corners in the rectangle can be depicted by a large *X*, this structure is known as an *X* Wing.

Alas, even after all that work the puzzle still will not fall. It is time to pull out the big guns.

1.5 ARIADNE'S THREAD

We are working hard for each cell, but it will soon be worth it. Look at the labeled cells below left ar the candidate values for those cells shown below right. We see that the cell labeled *A* contains either 3 or a 9, but we do not know which. Our strategy will be to make a preliminary guess as to which value is correct. We will then follow the logical consequences of that decision, in the hopes of thereb gaining some insight regarding the vacant cells.



Let us try placing a 3 in cell *A*. That forces cell *B* to contain a 7. Cell *C* is then forced to contain a 2, which forces cell *D* to be a 4. Then cell *E* contains a 9. As fun as this is, however, our luck is abou to run out. For now we must inquire as to the location of the 9 in the upper right block. Having place a 3 in cell *A*, and having been forced thereby to place a 9 in cell *E*, we are entirely out of options. It seems that our experiment of placing a 3 in cell *A* has led to a contradiction. But since that forces us to placea9in that cell we see that, after all our hard work, another cell has fallen.

In his essay on solving Sudoku puzzles [30], Michael Mepham refers to this style of thought as “Ariadne’s thread.” The reference is to Greek mythology. It seems that Ariadne’s lover, Theseus, ha entered the labyrinth of King Milos with the intention of killing the dreaded Minotaur. To keep him from getting lost, Ariadne gave Theseus a long, silken thread. Theseus unrolled the thread as he proceeded through the maze. Then, upon hitting a dead end, he could backtrack to the most recent fork and take a different path.

That is precisely what we have done. We followed the path of placing a 2 in cell *A* until it led us t a dead end. We rectified our error by backtracking to the point of our fallacious assumption and replaced it with a more reasonable choice.

You might object that Ariadne’s Thread hardly counts as a solving technique, since it seems tantamount to guessing. We would suggest, however, that this is not the best way of looking at thing In a proper Sudoku puzzle, the value in each cell is logically determined by the placement of the starting clues. A ‘solving technique’ is any method that aids you in discerning the relevant deductions. In some cases, as with a forced cell or a twin, the logic is straightforward and easy to see In others, like a triple or an *X*-Wing, more subtle reasoning is required. Regardless, in every case you are asking yourself, “What are the logical consequences of placing this digit in this cell?”’

So it is with Ariadne’s Thread. It is not that this technique involves guessing, whereas our other techniques do not. It differs from the other techniques only in the complexity of the deductions needed to make it work. This is not surprising. After all, Ariadne’s Thread is the technique to which you resort after the simpler methods have proven inadequate. In some especially difficult puzzles, th logical chain might be of such length and complexity that it defeats the abilities of all but the most skillful solvers. For all of that, however, it is not fundamentally different from our other solving techniques, and it is not comparable to outright guessing.

We have now forced our way through the roadblocks of this puzzle, and the rest falls in line fairly easily. You can finish up and join us after the jump.

1.6 ARE WE DOING MATH YET?

Suppose there is a barn containing cows and chickens, 50 animals in all. We notice that there are a total of 144 feet on the ground. Keeping in mind that chickens have 2 feet while cows have 4, can you determine the number of cows and chickens?

No doubt you took an algebra class at some point in your life, and if you did you might come up with an argument like this: Let x denote the number of cows. Then $50 - x$ denotes the number of chickens. Then the number of cow feet is $4x$ while the number of chicken feet is $2(50 - x)$. Since the total number of legs is 144, we have

$$2(50 - x) + 4x = 144$$

$$100 + 2x = 144$$

$$x = 22.$$

There must be 22 cows, and therefore 28 chickens, in the barn.

Now *that's* a math problem! We used algebra and everything.

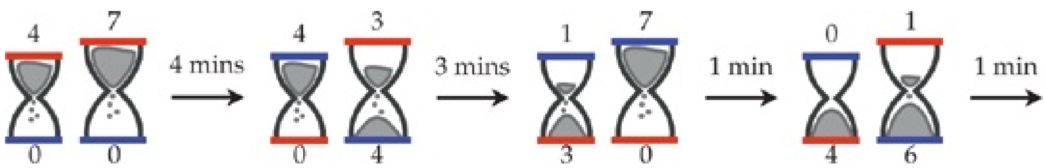
However, if using algebra were the criterion, then this would cease to be a math problem as soon a someone thinks of telling the cows to stand on their hind legs. There would then be fifty animals, eac with two feet on the ground, for a total of one hundred feet. That means there are forty-four feet in th air. Since each cow has two feet in the air, there must be twenty-two cows. Simple as that.

Attention, cows, please stand up

Yes, you might say, that is terribly clever. But we used arithmetic so it is still math.

Then what about the hourglass problem from the chapter's preamble? We were asked to time a period of nine minutes using only a four-minute and a seven-minute hourglass. Here is one possible solution. For convenience, we will refer to the four-minute glass as *F* and the seven-minute glass as *S* Begin by flipping over both of them. After four minutes, *F* is empty while *S* still has three minutes to go. Now flip over *F*. Three minutes later, after a total of seven minutes have elapsed, *S* is empty whil *F* has one minute left. Flip over *S*. One minute later *F* is empty, while *S* has one minute of sand in its base. Now flip *S* again. When it runs out exactly nine minutes have elapsed. Here it is in pictures:

Clever hourglass flipping to measure nine minutes



Was that a math problem? No arithmetic, really, just basic counting and a bit of cleverness. But you might still argue that numbers were involved, so it is still math.

Then what about the water and wine problem? Recall that we had a liter of water and a liter of wine. Some of the wine was transferred to the water bottle and mixed in thoroughly. Then enough of the water/wine mixture was transferred back to the wine bottle so that both containers again containe one liter of liquid. Is there now more wine in the water bottle or water in the wine bottle?

One of the charming aspects of this problem is that it is typical to go from complete befuddlemen to perfect understanding in an instant. There is no middle ground where you sort of see what is going