

STRING THEORY and the  
GEOMETRY of the UNIVERSE'S  
HIDDEN DIMENSIONS

THE

SHAPE OF

INNER SPACE



SHING-TUNG YAU

WINNER OF THE FIELDS MEDAL

and STEVE NADIS

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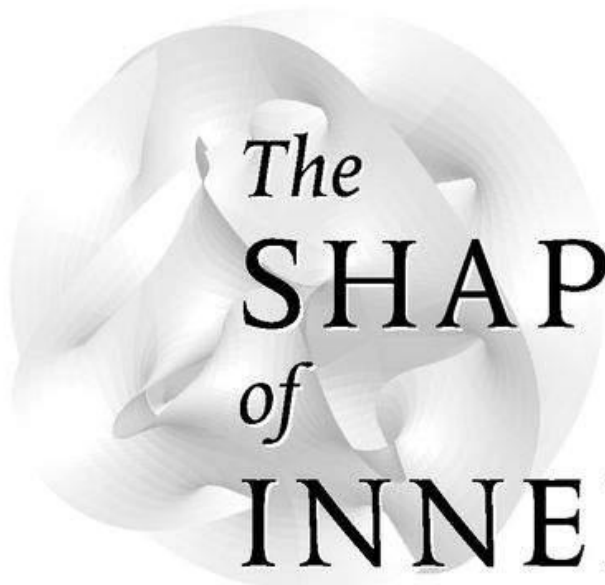
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*The*  
**SHAPE**  
*of*  
**INNER**  
**SPACE**

STRING THEORY AND THE  
GEOMETRY OF THE UNIVERSE'S  
HIDDEN DIMENSIONS

Shing-Tung Yau  
*and*  
Steve Nadis

*Illustrations by*  
*Xianfeng (David) Gu and Xiaotian (Tim) Yin*

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# SPACE/TIME

*Time, time  
why does it vanish?  
All manner of things  
what infinite variety.  
Three thousand rivers  
all from one source.  
Time, space  
mind, matter, reciprocal.  
Time, time  
it never returns.  
Space, space  
how much can it hold?  
In constant motion  
always in flux.  
Black holes lurking  
mysteries afoot.  
Space and time  
one without bounds.  
Infinite, infinite  
the secrets of the universe.  
Inexhaustible, lovely  
in every detail.  
Measure time, measure space  
no one can do it.  
Watched through a straw  
what's to be learned has no end.*

SHING-TUNG YAU  
BEIJING, 2002

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# PREFACE

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Mathematics is often called the language of science, or at least the language of the physical sciences, and that is certainly true: Our physical laws can only be stated precisely in terms of mathematical equations rather than through the written or spoken word. Yet regarding mathematics as merely a language doesn't do justice to the subject at all, as the word leaves the erroneous impression that, save for some minor tweaks here and there, the whole business has been pretty well sorted out.

In fact, nothing could be further from the truth. Although scholars have built a strong foundation over the course of hundreds—and indeed thousands—of years, mathematics is still very much a thriving and dynamic enterprise. Rather than being a static body of knowledge (not to suggest that languages themselves are set in stone), mathematics is actually a dynamic, evolving science, with new insights and discoveries made every day rivaling those made in other branches of science, though mathematical discoveries don't capture the headlines in the same way that the discovery of a new elementary particle, a new planet, or a new cure for cancer does. In fact, save for the proof of a centuries-old problem from time to time, they rarely capture headlines at all.

Yet for those who appreciate the sheer force of mathematics, it can be viewed as not just a language but as the surest path to the truth—the bedrock upon which the whole edifice of physical science rests. The strength of this discipline, again, lies not simply in its ability to explain physical reality or to reveal it, because to a mathematician, mathematics is reality. The geometric figures and spaces, whose existence we prove, are just as real to us as are the elementary particles of physics that make up a matter. But we consider mathematical structures even more fundamental than the particles of nature because mathematical structures can be used not only to understand such particles but also to understand the phenomena of everyday life, such as the contours of a human face or the symmetry of flowers. What excites geometers perhaps most of all is the power and beauty of the abstract principles that underlie the familiar forms and shapes of our contemporary world.

For me, the study of mathematics and my specialty, geometry, has truly been an adventure. I still recall the thrill I felt during my first year of graduate school, when—as a twenty-year-old fresh off the boat, so to speak—I first learned about Einstein's theory of gravity. I was struck by the notion that gravity and curvature could be regarded as one and the same, as I'd already become fascinated with curved surfaces during my undergraduate years in Hong Kong. Something about these shapes appealed to me on a visceral level. I don't know why, but I couldn't stop thinking about them. Hearing that curvature lay at the heart of Einstein's theory of general relativity gave me hope that someday, and in some way, I might be able to contribute to our understanding of the universe.

*The Shape of Inner Space* describes my explorations in the field of mathematics, focusing on one discovery in particular that has helped some scientists build models of the universe. No one can say for sure whether these models will ultimately prove correct. But the theory underlying these models nevertheless, possesses a beauty that I find undeniable.

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Taking on a book of this nature has been challenging, to say the least, for someone like me who is more comfortable with geometry and nonlinear differential equations than writing in the English language, which is not my native tongue. I find it frustrating because there's a great clarity, as well as a kind of elegance, in mathematical equations that is difficult, if not impossible, to express in words. It's a bit like trying to convey the majesty of Mount Everest or Niagara Falls without any pictures.

Fortunately, I've gotten some well-needed help on this front. Although this narrative is told through my eyes and in my voice, my coauthor has been responsible for translating the abstract and abstruse mathematics into (hopefully) lucid prose.

When I proved the Calabi conjecture—an effort that lies at the heart of this book—I dedicated the paper containing that proof to my late father, Chen Ying Chiu, an educator and philosopher who instilled in me a respect for the power of abstract thought. I dedicate this book to him and to my late mother, Leung Yeuk Lam, both of whom had a profound influence on my intellectual growth. In addition, I want to pay tribute to my wife, Yu-Yun, who has been so tolerant of my rather excessive (and perhaps obsessive) research and travel schedule, and to my sons, Isaac and Michael, of whom I'm very proud.

I also dedicate this book to Eugenio Calabi, the author of the aforementioned conjecture, whom I've known for nearly forty years. Calabi was an enormously original mathematician with whom I've been linked for more than a quarter century through a class of geometric objects, Calabi-Yau manifolds, which serve as the principal subject of this book. The term *Calabi-Yau* has been so widely used since it was coined in 1984 that I almost feel as if Calabi is my first name. And if it is to be my first name—at least in the public's mind—it's one I'm proud to have.

The work that I do, much of which takes place along the borders between mathematics and theoretical physics, is rarely done in isolation, and I have benefited greatly from interactions with friends and colleagues. I'll mention a few people, among many, who have collaborated with me directly or inspired me in various ways.

First, I'd like to pay tribute to my teachers and mentors, a long line of illustrious people that includes S. S. Chern, Charles Morrey, Blaine Lawson, Isadore Singer, Louis Nirenberg, and the aforementioned Calabi. I'm pleased that Singer invited Robert Geroch to speak at a 1973 Stanford conference, where Geroch inspired my work with Richard Schoen on the positive mass conjecture. My subsequent interest in physics-related mathematics has always been encouraged by Singer.

I'm grateful for the conversations I had on general relativity while visiting Stephen Hawking and Gary Gibbons at Cambridge University. I learned about quantum field theory from one of the masters of the subject, David Gross. I remember in 1981, when I was a professor at the Institute for Advanced Study, the time Freeman Dyson brought a fellow physicist, who had just arrived in Princeton, into my office. The newcomer, Edward Witten, told me about his soon-to-be-published proof of the positive

mass conjecture—a conjecture I had previously proved with a colleague using a very different technique. I was struck, for the first of many times to come, by the sheer force of Witten mathematics.

Over the years, I've enjoyed close collaborations with a number of people, including Schoen (mentioned above), S. Y. Cheng, Richard Hamilton, Peter Li, Bill Meeks, Leon Simon, and Karen Uhlenbeck. Other friends and colleagues who have added to this adventure in many ways include Simon Donaldson, Robert Greene, Robert Osserman, Duong Hong Phong, and Hung-Hsi Wu.

I consider myself especially lucky to have spent the past twenty-plus years at Harvard, which has provided an ideal environment for interactions with both mathematicians and physicists. During my time here, I've gained many insights from talking to Harvard math colleagues—such as Joseph Bernstein, Noam Elkies, Dennis Gaitsgory, Dick Gross, Joe Harris, Heisuke Hironaka, Arthur Jaffe (also a physicist), David Kazhdan, Peter Kronheimer, Barry Mazur, Curtis McMullen, David Mumford, Wilfried Schmid, Yum-Tong Siu, Shlomo Sternberg, John Tate, Cliff Taubes, Richard Taylor, H. T. Yau, and the late Raoul Bott and George Mackey—while having memorable exchanges with MIT math colleagues as well. On the physics side, I've had countless rewarding conversations with Andy Strominger and Cumrun Vafa.

In the past ten years, I was twice an Eilenberg visiting professor at Columbia, where I had many stimulating conversations with faculty members, especially with Dorian Goldfeld, Richard Hamilton, Duong Hong Phong, and S. W. Zhang. I was also a Fairchild visiting professor and Moore visiting professor at Caltech, where I learned a lot from Kip Thorne and John Schwarz.

Over the last twenty-three years, I have been supported by the U.S. government through the National Science Foundation, the Department of Energy, and DARPA in my research related to physics. Most of my postdoctoral fellows received their Ph.D.s in physics, which is somewhat unusual in our discipline of mathematics. But the arrangement has been mutually beneficial, as they have learned some mathematics from me and I have learned some physics from them. I am glad that many of these postdoctoral fellows with physics backgrounds later became prominent professors in mathematics departments at Brandeis, Columbia, Northwestern, Oxford, Tokyo, and other universities. Some of my postdocs have done important work on Calabi-Yau manifolds, and many of them have also helped on this book: Mboyo Esole, Brian Greene, Gary Horowitz, Shinobu Hosono, Tristan Hubsch, Albrecht Klemm, Bong Lian, James Sparks, Li-Sheng Tseng, Satoshi Yamaguchi, and Eric Zaslow. Finally, my former graduate students—including Jun Li, Kefeng Liu, Melissa Li, Dragon Wang, and Mu-Tao Wang—have made noteworthy contributions in this area as well, some of which will be described in the pages to come.

—SHING-TUNG YAU, CAMBRIDGE, MASSACHUSETTS, MARCH 2010

Odds are I never would have known about this project were it not for Henry Tye, a Cornell physicist (and a friend of Yau's), who suggested that my coauthor-to-be might steer me to an interesting tale of two. Henry was right about this, as he has been about many other things. I'm grateful to him for helping to launch me on this unexpected journey and for assisting me at many junctures along the way.

As Yau has often said, when you venture down a path in mathematics, you never know where it will end up. The same has been true on the writing end of things. The two of us pretty much agreed during our very first meeting to write a book together, though it took a long while for us to know what the book would be about. In some ways, you might say we didn't really know that until the book was finished.

Now a few words about the product of this collaboration in an attempt to keep any confusion to a minimum. My coauthor is, of course, a mathematician whose work is central to much of the story related here. Sections of the book in which he was an active participant are generally written in the first person, with the "I" in this case referring to him and him alone. However, even though the book has its fair share of personal narrative, this work should probably not be characterized as Yau's autobiography or biography. That's because part of the discussion relates to people Yau doesn't know (or who died long before he was born), and some of the subject matter described—such as experimental physics and cosmology—lies outside his areas of expertise. These sections, which are written in a third-person voice, are largely based on interviews and other research I conducted.

While the book is, admittedly, an unusual blend of our different backgrounds and perspectives, it seemed to be the best way for the two of us to recount a story that we both considered worth telling. The task of actually getting this tale down on paper relied heavily on my coauthor's extraordinary grasp of numbers and hopefully profited as well from his collaborator's facility with words.

One other point on the issue of whether this ought to be regarded as an autobiography: Although the book certainly revolves around Yau's work, I would suggest that the main character is not Yau himself but rather the class of geometric shapes—so-called Calabi-Yau manifolds—that he helped invent.

Broadly speaking, this book is about understanding the universe through geometry. General relativity, a geometry-based description of gravity that has achieved stunning success in the past century, offers one example. String theory represents an ambitious attempt to go even further, and geometry is vital to this quest, with six-dimensional Calabi-Yau shapes assuming a special place in this theory. The book tries to present some of the ideas from geometry and physics needed to understand where Calabi-Yau manifolds came from and why some physicists and mathematicians consider them important. The book focuses on various aspects of these manifolds—their defining features, the mathematics that led to their discovery, the reasons string theorists find them intriguing, and the question of whether these shapes hold the key to our universe (and perhaps to other universes as well).



That, at least, is what *The Shape of Inner Space* is supposed to be about. Whether it lives up to the billing may be open to debate. But there is no doubt in my mind that this book would never have come to fruition without technical, editorial, and emotional support from many people—too many, I’m afraid, to list in full, but I will mention as many as I can.

I received a tremendous amount of help from people already singled out by my coauthor. They include Eugenio Calabi, Simon Donaldson, Brian Greene, Tristan Hubsch, Andrew Strominger, Li Sheng Tseng, Cumrun Vafa, Edward Witten, and, most of all, Robert Greene, Bong Lian, and Li-Sher Tseng. The latter three provided me with math and physics tutorials throughout the writing process exhibiting expository skills and levels of patience that boggle the mind. Robert Greene, in particular, spoke with me a couple of times a week during busy stretches to guide me through thorny bits of differential geometry. Without him, I would have been sunk—many times over. Lian got me started in thinking about geometric analysis, and Tseng helped out immensely with last-minute changes in our ever-evolving manuscript.

The physicists Allan Adams, Chris Beasley, Shamit Kachru, Liam McAllister, and Burt Ovrut fielded questions from me at various times of day and night, carrying me through many a rough patch. Other individuals who were exceedingly generous with their time include Paul Aspinwall, Melan Becker, Lydia Bieri, Volker Braun, David Cox, Frederik Denef, Robbert Dijkgraaf, Ron Donagi, Mike Douglas, Steve Giddings, Mark Gross, Arthur Hebecker, Petr Horava, Matt Kleban, Igor Klebanov, Albion Lawrence, Andrei Linde, Juan Maldacena, Dave Morrison, Lubos Motl, Hiroshi Ooguri, Tom Pantev, Ronen Plesser, Joe Polchinski, Gary Shui, Aaron Simons, Raman Sundrum, Wati Taylor, Brent Underwood, Deane Yang, and Xi Yin.

That is merely the tip of the iceberg, as I’ve also received help from Eric Adelberger, Saleem Al-Bruce Allen, Nima Arkani-Hamed, Michael Atiyah, John Baez, Thomas Banchoff, Katrin Becker, George Bergman, Vincent Bouchard, Philip Candelas, John Coates, Andrea Cross, Lance Dixon, David Durlach, Dirk Ferus, Felix Finster, Dan Freed, Ben Freivogel, Andrew Frey, Andreas Gathmann, Doron Gepner, Robert Geroch, Susan Gilbert, Cameron Gordon, Michael Green, Paul Green, Arthur Greenspoon, Marcus Grisaru, Dick Gross, Monica Guica, Sergei Gukov, Alan Guth, Robert S. Harriet, Matt Headrick, Jonathan Heckman, Dan Hooper, Gary Horowitz, Stanislaw Janeczko, Lizhen Jiang, Sheldon Katz, Steve Kleiman, Max Kreuzer, Peter Kronheimer, Mary Levin, Avi Loeb, Feng Luo, Erwin Lutwak, Joe Lykken, Barry Mazur, William McCallum, John McGreevy, Stephen Miller, Clifford Moore, Steve Nahn, Gail Oskin, Rahul Pandharipande, Joaquín Pérez, Roger Penrose, Miles Reid, Nicolai Reshetikhin, Kirill Saraikin, Karen Schaffner, Michael Schulz, John Schwarz, Ashoke Sen, Kris Snibbe, Paul Shellard, Eva Silverstein, Joel Smoller, Steve Strogatz, Leonard Susskind, Yaakov Soibelman, Erik Swanson, Max Tegmark, Ravi Vakil, Fernando Rodriguez Villegas, Dwight Vincent, Dan Waldram, Devin Walker, Brian Wecht, Toby Wiseman, Jeff Wu, Chen Ning Yang, Donald Zeyl, and others.

Many of the concepts in this book are difficult to illustrate, and we were fortunate to be able to draw on the extraordinary graphic talents of Xiaotian (Tim) Yin and Xianfeng (David) Gu of the Stony Brook Computer Science Department, who were assisted in turn by Huayong Li and Wei Zeng. Additional help on the graphics front was provided by Andrew Hanson (the premier renderer of

Calabi-Yau manifolds), John Oprea, and Richard Palais, among others.

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I thank my many friends and relatives, including Will Blanchard, John De Lancey, Ross Eatman, Evan Hadingham, Harris McCarter, and John Tibbetts, who read drafts of the book proposal and chapters or otherwise offered advice and encouragement along the way. Both my coauthor and I are grateful for the invaluable administrative assistance provided by Maureen Armstrong, Lily Chan, Haoyang Xu, and Gena Bursan.

Several books proved to be valuable references. Among them are *The Elegant Universe* by Brian Greene, *Euclid's Window* by Leonard Mlodinow, *Poetry of the Universe* by Robert Osserman, and *The Cosmic Landscape* by Leonard Susskind.

*The Shape of Inner Space* might never have gotten off the ground were it not for the help of John Brockman, Katinka Matson, Michael Healey, Max Brockman, Russell Weinberger, and others at the Brockman, Inc., literary agency. T. J. Kelleher of Basic Books had faith in our manuscript when others did not, and—with the help of his colleague, Whitney Casser—worked hard to get our book into presentable form. Kay Mariea, the project editor at Basic Books, shepherded our manuscript through its many stages, and Patricia Boyd provided expert copyediting, teaching me that “the same” and “exactly the same” are exactly the same thing.

Finally, I’m especially grateful for the support from my family members—Melissa, Juliet, and Pauline, along with my parents Lorraine and Marty, my brother Fred, and my sister Sue—who acted as if six-dimensional Calabi-Yau manifolds were the most fascinating thing in the world, not realizing that these manifolds are, in fact, out of this world.

—STEVE NADIS, CAMBRIDGE, MASSACHUSETTS, MARCH 2010

## THE SHAPES OF THINGS TO COME

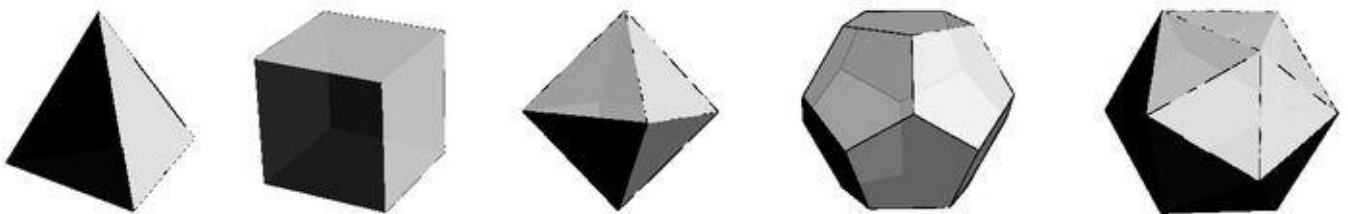
*God ever geometrizes.*

—PLATO

In the year 360 B.C. or thereabouts, Plato published *Timaeus*—a creation story told in the form of a dialogue between his mentor, Socrates, and three others: Timaeus, Hermocrates, and Critias. Timaeus is likely a fictitious character who is said to have come to Athens from the southern Italian city of Locri. He is an “expert in astronomy [who] has made it his main business to know the nature of the universe.” Through Timaeus, Plato presents his own theory of everything, with geometry playing a central role in those ideas.

Plato was particularly fascinated with a group of convex shapes, a special class of polyhedra that have since come to be known as the Platonic solids. The faces of each solid consist of identical polygons. The tetrahedron, for example, has four faces, each a triangle. The hexahedron, or cube, is made up of six squares. The octahedron consists of eight triangles, the dodecahedron of twelve pentagons, and the icosahedron of twenty triangles.

Plato did not invent the solids that bear his name, and no one knows who did. It is generally believed, however, that one of his contemporaries, Theaetetus, was the first to prove that there are only five, such solids—or *convex regular polyhedra*, as they’re called—exist. Euclid gave a complete mathematical description of these geometric forms in *The Elements*.



0.1—The five *Platonic solids*, named for the Greek philosopher Plato: the tetrahedron, hexahedron (cube), octahedron, dodecahedron, and icosahedron. The prefixes derive from the number of faces: four, six, eight, twelve, and twenty, respectively. One feature of these solids that no other convex polyhedra satisfy is that all their faces, edges, and angles (between two edges) are congruent.

The Platonic solids have several intriguing properties, some of which turn out to be equivalent ways of describing them. For each type of solid, the same number of faces meet at each of the corner points or vertices. One can draw a sphere around the solid that touches every one of those vertices—something that’s not possible for polyhedra in general. Moreover, the angle of each vertex, where two edges meet, is always the same. The number of vertices plus faces equals the number of edges plus two.

Plato attached a metaphysical significance to the solids, which is why his name is forever linked with them. In fact, the convex regular polyhedra, as detailed in *Timaeus*, formed the very essence of his cosmology. In Plato’s grand scheme of things, there are four basic elements: earth, air, fire, and water. If we could examine these elements in fine detail, we’d notice that they are composed of minuscule versions of the Platonic solids: Earth would thus consist of tiny cubes, the air of octahedrons, fire of tetrahedrons, and water of icosahedrons. “One other construction, a fifth, still remained,” Plato wrote in *Timaeus*, referring to the dodecahedron. “And this one god used for the whole universe, embroidering figures on it.”<sup>2</sup>

As seen today, with the benefit of 2,000-plus years of science, Plato’s conjecture looks rather dubious. While there is, at present, no ironclad agreement as to the basic building blocks of the universe—be they leptons and quarks, or hypothetical subquarks called preons, or equally hypothetical strings—we do know that it’s not just earth, air, fire, and water embroidered upon a giant dodecahedron. Nor do we believe that the properties of the elements are governed strictly by the shapes of Platonic solids.

On the other hand, Plato never claimed to have arrived at the definitive theory of nature. He considered *Timaeus* a “likely account,” the best he could come up with at the time, while conceding that others who came after him might very well improve on the picture, perhaps in a dramatic way. As *Timaeus* states midway into his discourse: “If anyone puts this claim to the test and discovers that it isn’t so, his be the prize, with our congratulations.”<sup>3</sup>

There’s no question that Plato got many things wrong, but viewing his thesis in the broadest sense it’s clear that he got some things right as well. The eminent philosopher showed perhaps the greatest wisdom in acknowledging that what he put forth might not be true, but that another theory, perhaps building on some of his ideas, could be true. The solids, for instance, are objects of extraordinary symmetry: The icosahedron and dodecahedron, for instance, can be rotated sixty ways (which, not coincidentally, turns out to be twice the number of edges in each shape) and still look the same. In basing his cosmology on these shapes, Plato correctly surmised that symmetry ought to lie at the heart of any credible description of nature. For if we are ever to produce a real theory of everything—which all the forces are unified and all the constituents obey a handful (or two) of rules—we’ll need to uncover the underlying symmetry, the simplifying principle from which everything else springs.

It hardly bears mentioning that the symmetry of the solids is a direct consequence of their precise shape or geometry. And this is where Plato made his second big contribution: In addition to realizing that mathematics was the key to fathoming our universe, he introduced an approach we now call the

geometrization of physics—the same leap that Einstein made. In an act of great prescience, Poincaré suggested that the elements of nature, their qualities, and the forces that act upon them may all be the result of some hidden geometrical structure that conducts its business behind the scenes. The world we see, in other words, is a mere reflection of the underlying geometry that we might not see. This is a notion dear to my heart, and it relates closely to the mathematical proof for which I am best known—to the extent that I am known at all. Though it may strike some as far-fetched, yet another case of geometric grandstanding, there just might be something to this idea, as we'll see in the pages ahead.

## A UNIVERSE IN THE MARGINS

The invention of the telescope, and its steady improvement over the years, helped confirm what had become a truism: There's more to the universe than we can see. Indeed, the best available evidence suggests that nearly three-fourths of all the stuff of the cosmos lies in a mysterious, invisible form called dark energy. Most of the rest—excluding only the 4 percent composed of ordinary matter that includes us—is called dark matter. And true to form, it too has proved “dark” in just about every respect: hard to see and equally hard to fathom.

The portion of the cosmos we can see forms a sphere with a radius of about 13.7 billion light-years. This sphere is sometimes referred to as a Hubble volume, but no one believes that's the full extent of the universe. According to the best current data, the universe appears to extend limitlessly, with straight lines literally stretching from here to eternity in every direction we can point.

There's a chance, however, that the universe is ultimately curved and bounded. But even if it is, the allowable curvature is so slight that, according to some analyses, the Hubble volume we see is just one out of at least one thousand such volumes that must exist. And a recently launched space instrument—the Planck telescope, may reveal within a few years that there are at least *one million* Hubble volumes out there in the cosmos, only one of which we'll ever have access to.<sup>1</sup> I'm trusting the astrophysicists on this one, realizing that some may quarrel with the exact numbers cited above. One fact, however, appears to be unassailable: What we see is just the tip of the iceberg.

At the other extreme, microscopes, particle accelerators, and various imaging devices continue to reveal the universe on a miniature scale, illuminating a previously inaccessible world of cells, molecules, atoms, and smaller entities. By now, none of this should be all that surprising. We fully expect our telescopes to probe ever deeper into space, just as our microscopes and other tools bring more of the invisible to light.

But in the last few decades—owing to developments in theoretical physics, plus some advances in geometry that I've been fortunate enough to participate in—there has been another realization that's even more startling: Not only is there more to the universe than we can see, but there may even be more dimensions, and possibly quite a few more than the three spatial dimensions we're intimately acquainted with.

That's a tough proposition to swallow, because if there's one thing we know about our world—there's one thing our senses have told us from our first conscious moments and first groping

explorations—it's the number of dimensions. And that number is three. Not three, give or take a dimension or so, but exactly three. Or so it seemed for the longest time. But maybe, just maybe, there are additional dimensions so small that we haven't noticed them yet. And despite their modest size, they could be crucial in ways we could not have possibly appreciated from our entrenched, three-dimensional perspective.

While this may be hard to accept, we've learned in the past century that whenever we stray far from the realm of everyday experience, our intuition can fail us. If we travel extremely fast, special relativity tells us that time slows down, not something you're likely to intuit from common sense. If we make an object extremely small, according to the dictates of quantum mechanics, we can't say exactly where it is. When we do experiments to determine whether the object has ended up behind Door A or Door B, we find it's neither here nor there, in the sense that it has no absolute position. (And it sometimes may appear to be in both places at once!) Strange things, in other words, can and will happen, and it's possible that tiny, hidden dimensions are one of them.

If this idea is true, then there might be a kind of universe in the margins—a critical chunk of real estate tucked off to the side, just beyond the reach of our senses. This would be revolutionary in two ways. The mere existence of extra dimensions—a staple of science fiction for more than a hundred years—would be startling enough on its own, surely ranking among the greatest findings in the history of physics. But such a discovery would really be a starting point rather than an end unto itself. For just as a general might obtain a clearer perspective on the battlefield by observing the proceedings from a hilltop or tower and thereby gaining the benefit of a vertical dimension, so too may our laws of physics become more apparent, and hence more readily discerned, when viewed from a higher-dimensional vantage point.

We're familiar with travel in three basic directions: north or south, east or west, and up or down. (Or equivalently, left or right, backward or forward, and up or down.) Wherever we go—whether it's driving to the grocery store or flying to Tahiti—we move in some combination of those three independent directions. So familiar are we with these dimensions that trying to conceive of a fourth additional dimension—and figuring out exactly where it would point—might seem impossible. For a long while, it seemed as if what you see is what you get. In fact, more than two thousand years ago Aristotle argued as much in his treatise *On the Heavens*: “A magnitude if divisible one way is a line, if two ways a surface, and if three a body. Beyond these there is no other magnitude, because the three dimensions are all that there are.”<sup>2</sup> In A.D. 150, the astronomer and mathematician Ptolemy tried to prove that four dimensions are impossible, insisting that you cannot draw four mutually perpendicular lines. A fourth perpendicular, he contended, would be “entirely without measure and without definition.”<sup>3</sup> His argument, however, was less a rigorous proof than a reflection of our inability both to visualize and to draw in four dimensions.

To a mathematician, a dimension is a “degree of freedom”—an independent way of moving through space. A fly buzzing around over our heads is free to move in any direction the skies permit. Assuming there are no obstacles, it has three degrees of freedom. Suppose that fly lands on a parking lot and gets stuck in a patch of fresh tar. While it is temporarily immobilized, the fly has zero degrees

of freedom and is effectively confined to a single spot—a zero-dimensional world. But this creature persistent and, after some struggle, wrests itself free from the tar, though injuring its wing in the process. Unable to fly, it has two degrees of freedom and can roam the surface of the parking lot will. Sensing a predator—a ravenous frog, perhaps—our hero seeks refuge in a rusted tailpipe lying the lot. The fly thus has one degree of freedom, trapped at least for now in the one-dimensional linear world of this narrow pipe.

But is that all there is? Does a fly buzzing through the air, stuck in tar, crawling on the asphalt, or making its way through a pipe include all the possibilities imaginable? Aristotle or Ptolemy would have said yes, but while this may be the case for a not terribly enterprising fly, it is not the end of the story for contemporary mathematicians, who typically find no compelling reason to stop at three dimensions. On the contrary, we believe that to truly understand a concept in geometry, such as curvature or distance, we need to understand it in all possible dimensions, from zero to  $n$ , where  $n$  can be a very big number indeed. Our grasp of that concept will be incomplete if we stop at three dimensions—the point being that if a rule or law of nature works in a space of any dimension, it is more powerful, and seemingly more fundamental, than a statement that only applies in a particular setting.

Even if the problem you're grappling with pertains to just two or three dimensions, you might still secure helpful clues by studying it in a variety of dimensions. Let's return to our example of the fly flitting about in three-dimensional space, which has three directions in which to move, or three degrees of freedom. Yet let's suppose another fly is moving freely in that same space; it too has three degrees of freedom, and the system as a whole has suddenly gone from three to six dimensions—with six independent ways of moving. With more flies zigzagging through the space—all moving on their own without regard to the other—the complexity of the system goes up, as does the dimensionality.

One advantage in looking at higher-dimensional systems is that we can divine patterns that might be impossible to perceive in a simpler setting. In the next chapter, for instance, we'll discuss the fact that on a spherical planet, hypothetically covered by a giant ocean, all the water cannot flow in the same direction—say, from west to east—at every point. There have to be some spots where the water is not moving at all. Although this rule applies to a two-dimensional surface, it can only be derived by looking at a much higher-dimensional system in which all possible configurations—all possible movements of tiny bits of water on the surface—are considered. That's why we continually push to higher dimensions to see what it might lead to and what we might learn.

One thing that higher dimensions lead to is greater complexity. In topology, which classifies objects in terms of shape in the most general sense, there are just two kinds of one-dimensional spaces: a line (a curve with two open ends) and a circle (a closed curve with no ends). There aren't any other possibilities. Of course, the line could be squiggly, or the closed curve oblong, but those are questions of geometry, not topology. The difference between geometry and topology is like the difference between looking at the earth's surface with a magnifying glass and going up in a rocket ship and surveying the planet as a whole. The choice comes down to this: Do you insist on knowing every last detail—every ridge, undulation, and crevice in the surface—or will the big picture (“a giant round ball”) suffice? Whereas geometers are often concerned with identifying the exact shape and curvature

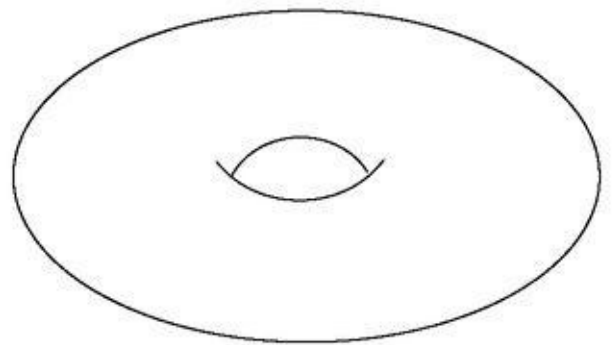
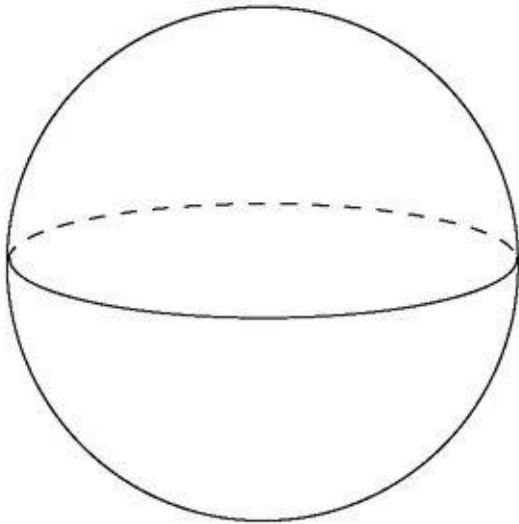
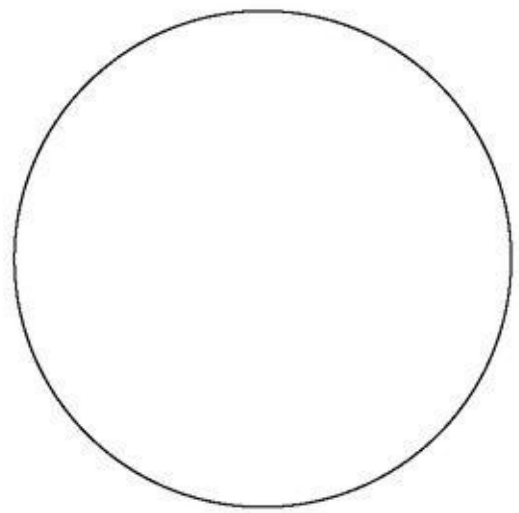


of some object, topologists only care about the overall shape. In that sense, topology is a holistic discipline, which stands in sharp contrast to other areas of mathematics in which advances are typically made by taking complicated objects and breaking them down into smaller and simpler pieces.

As for how this ties into our discussion of dimensions, there are—as we’ve said—just two basic one-dimensional shapes in topology: A straight line is identical to a wiggly line, and a circle is identical to any “loop”—oblong, squiggly, or even square—that you can imagine. The number of two-dimensional spaces is similarly restricted to two basic types: either a sphere or a donut. A topologist considers any two-dimensional surface without holes in it to be a sphere, and this includes everyday geometric shapes such as cubes, prisms, pyramids, and even watermelon-like objects called ellipsoids.

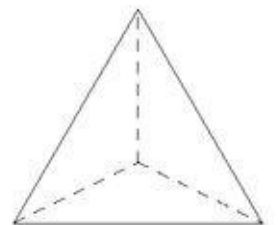
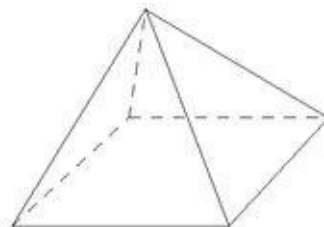
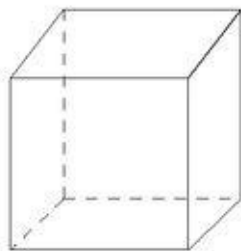
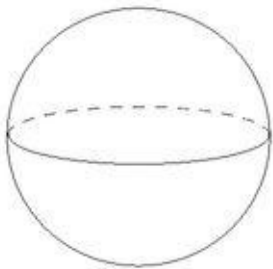
The presence of the hole in the donut or the lack of the hole in the sphere makes all the difference in this case: No matter how much you manipulate or deform a sphere—without ripping a hole in it, that is—you’ll never wind up with a donut, and vice versa. In other words, you cannot create new holes in an object, or otherwise tear it, without changing its topology. Conversely, topologists regard two shapes as functionally equivalent if—supposing they are made out of malleable clay or Play-Doh—one shape can be molded into the other by squeezing and stretching but not ripping.

A donut with one hole is technically called a *torus*, but a donut-like surface could have any number of holes. Two-dimensional surfaces that are both compact (closed up and finite in extent) and orientable (double-sided) can be classified by the number of holes they have, which is also known as their *genus*. Objects that look quite different in two dimensions are considered topologically identical if they have the same genus.

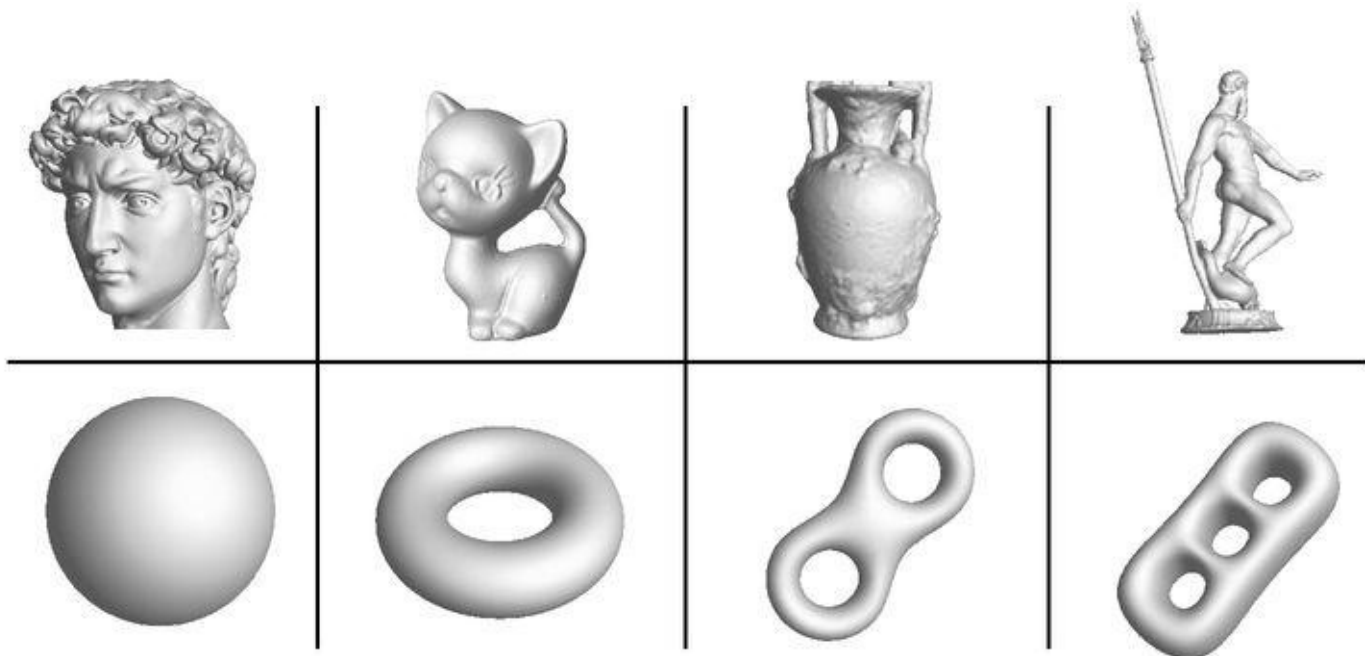


1.1—In topology, there are just two kinds of one-dimensional spaces that are fundamentally distinct from each other: a line and a circle. You can make a circle into all kinds of loops, but you can't turn a circle into a line without cutting it.

Two-dimensional surfaces, which are *orientable*—meaning they have two sides like a beach ball rather than just one side like a Möbius strip—can be classified by their *genus*, which can be thought of, in simple terms, as the number of holes. A sphere of genus 0, which has no holes, is therefore fundamentally distinct from a donut of genus 1, which has one hole. As with the circle and line, you can't transform a sphere into a donut without cutting a hole through the middle of it.



1.2—In topology, a sphere, cube, and tetrahedron—among other shapes—are all considered equivalent because each can be fashioned from the other by bending, stretching, or pushing, without their having to be torn or cut.



1.3—Surfaces of genus 0, 1, 2, and 3; the term *genus* refers to the number of holes.

The point made earlier about there being just two possible two-dimensional shapes—a donut or sphere—is only true if we restrict ourselves to orientable surfaces, and those are the surfaces we generally be referring to in this book. A beach ball, for example, has two sides, an inside and an outside, and the same goes for a tire’s innertube. There are, however, more complicated cases—single-sided, “nonorientable” surfaces such as the Klein bottle and Möbius strip—where the foregoing is not true.

In dimensions three and beyond, the number of possible shapes widens dramatically. In contemplating higher-dimensional spaces, we must allow for movements in directions we can’t readily imagine. We’re not talking about heading somewhere in between north and west like northwest or even “North by Northwest.” We’re talking about heading off the grid altogether, following arrows in a coordinate system that has yet to be drawn.

One of the first big breakthroughs in charting higher-dimensional space came courtesy of René Descartes, the seventeenth-century French mathematician, philosopher, scientist, and writer, though his work in geometry stands foremost for me. Among other contributions, Descartes taught us that thinking in terms of coordinates rather than pictures can be extremely productive. The labeling system he invented, which is now called the Cartesian coordinate system, united algebra and geometry. In a narrow sense, Descartes showed that by drawing  $x$ ,  $y$ , and  $z$  axes that intersect in a point and are all perpendicular to each other, one can pin down any spot in three-dimensional space with just three numbers—the  $x$ ,  $y$ , and  $z$  coordinates. But his contribution was much broader than that, as he vastly enlarged the scope of geometry and did so in one brilliant stroke. For with his coordinate system in hand, it became possible to use algebraic equations to describe complex, higher-dimensional geometric figures that are not readily visualized.

Using this approach, you can think about any dimension you want—not just  $(x, y, z)$  but  $(a, b, c, d, e, f)$  or  $(j, k, l, m, n, o, p, q, r, s)$ —the dimension of a given space being the number of coordinates you need to determine the location of a given point. Armed with this system, one could contemplate higher-dimensional spaces of any order—and do various calculations concerning them—without having to worry about trying to draw them.

The great German mathematician Georg Friedrich Bernhard Riemann took off with this idea two centuries later and carried it far. In the 1850s, while working on the geometry of curved (non-Euclidean) spaces—a subject that will be taken up in the next chapter—Riemann realized that these spaces were not restricted in terms of the number of dimensions. He showed how distance, curvature, and other properties in such spaces could be precisely computed. And in an 1854 inaugural lecture in which he presented principles that have since come to be known as Riemannian geometry, he speculated on the dimensionality and geometry of the universe itself. While still in his twenties, Riemann also began work on a mathematical theory that attempted to tie together electricity, magnetism, light, and gravity—thereby anticipating a task that continues to occupy scientists to this day.

Although Riemann helped free up space from the limitations of Euclidean flatness and three dimensions, physicists did not do much with that idea for decades. Their lack of interest may have stemmed from the absence of experimental evidence to suggest that space was curved or that any dimensions beyond three existed. What it came down to was that Riemann's advanced mathematics had simply outpaced the physics of his era, and it would take time—another fifty years or so—for the physicists, or at least one physicist in particular, to catch up. The one who did was Albert Einstein.

In developing his special theory of relativity—which was first presented in 1905 and further advanced in the years after, culminating in the general theory of relativity—Einstein drew on an idea that was also being explored by the German mathematician Hermann Minkowski, namely, that time is inextricably intertwined with the three dimensions of space, forming a new geometrical construction known as *spacetime*. In an unexpected turn, time itself came to be seen as the fourth dimension that Riemann had incorporated decades before in his elegant equations.

Curiously, the British writer H. G. Wells had anticipated this same outcome ten years earlier in his novel *The Time Machine*. As explained by the Time Traveller, the main character of that book, “there are really four dimensions, three which we call the three planes of Space, and a fourth, Time. There is, however, a tendency to draw an unreal distinction between the former three dimensions and the latter.”<sup>4</sup>

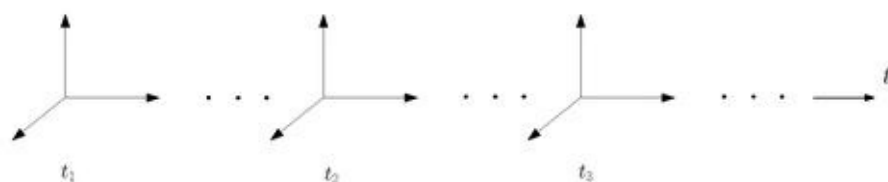
Minkowski said pretty much the same thing in a 1908 speech—except that in this case, he had the mathematics to back up such an outrageous claim: “Henceforth space by itself, and time by itself, are doomed to fade away into mere shadows, and only a kind of union of the two will preserve a *independent reality*.”<sup>5</sup> The rationale behind the marriage of these two concepts—if, indeed, marriages ever have a rationale—is that an object moves not only through space but through time as well. It thus takes four coordinates, three of space and one of time, to describe an event in four-dimensional

spacetime  $(x, y, z, t)$ .

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Although the idea may seem slightly intimidating, it can be expressed in extremely mundane terms. Suppose you make plans to meet somebody at a shopping mall. You note the location of the building—say it's at the corner of First Street and Second Avenue—and decide to meet on the third floor. That takes care of your  $x$ ,  $y$ , and  $z$  coordinates. Now all that remains is to fix the fourth coordinate and settle on the time. With those four pieces of information specified, your assignment is all set, barring any unforeseen circumstances that might intervene. But if you want to put it in Einstein's terms, you shouldn't look at it as setting the exact place for this little get-together, while separately agreeing on the time. What you're really spelling out is the location of this event in spacetime itself.

So in a single bound, early in the twentieth century, our conception of space grew from the cozy three-dimensional nook that had nurtured humankind since antiquity to the more esoteric realm of four-dimensional spacetime. This conception of spacetime formed the bedrock on which Einstein's theory of gravity, the general theory of relativity, was soon built. But is that the end of the line, as we asked once before? Does the buck stop there, at four dimensions, or can our notion of spacetime grow further still? In 1919, a possible answer to that question arrived unexpectedly in the form of a manuscript sent to Einstein for review by a then-unknown German mathematician, Theodor Kaluza.



1.4—As we don't know how to draw a picture in four dimensions, this is a rather crude, conceptual rendering of four-dimensional *spacetime*. The basic idea of spacetime is that the three spatial dimensions of our world (represented here by the  $x$ - $y$ - $z$  coordinate axis) have essentially the same status as a fourth dimension—that being time. We think of time as a *continuous variable* that's always changing, and the figure shows snapshots of the coordinate axis at various moments frozen in time:  $t_1$ ,  $t_2$ ,  $t_3$ , and so forth. In this way, we're trying to show that there are four dimensions overall: three of space plus the additional one labeled by time.

In Einstein's theory, it takes ten numbers—or ten *fields*—to precisely describe the workings of gravity in four dimensions. The force can be represented most succinctly by taking those ten numbers and arranging them in a four-by-four matrix more formally known as a *metric tensor*—a square table of numbers that serves as a higher-dimensional analogue of a ruler. In this case, the metric has sixteen entries in all, only ten of which are independent. Six of the numbers repeat because gravity, along with the other fundamental forces, is inherently symmetrical.

In his paper, Kaluza had basically taken Einstein's general theory of relativity and added an extra dimension to it by expanding the four-by-four matrix to a five-by-five one. By expanding spacetime to the fifth dimension, Kaluza was able to take the two forces known at the time, gravity and electromagnetism, and combine them into a single, unified force. To an observer in the five-

dimensional world that Kaluza envisioned, those forces would be one and the same, which is what we mean by unification. But in a four-dimensional world, the two can't go together; they would appear to be wholly autonomous. You could say that's the case simply because both forces do not fit into the same four-by-four matrix. The additional dimension, however, provides enough extra elbow room for both of them to occupy the same matrix and hence be part of the same, more all-encompassing force.

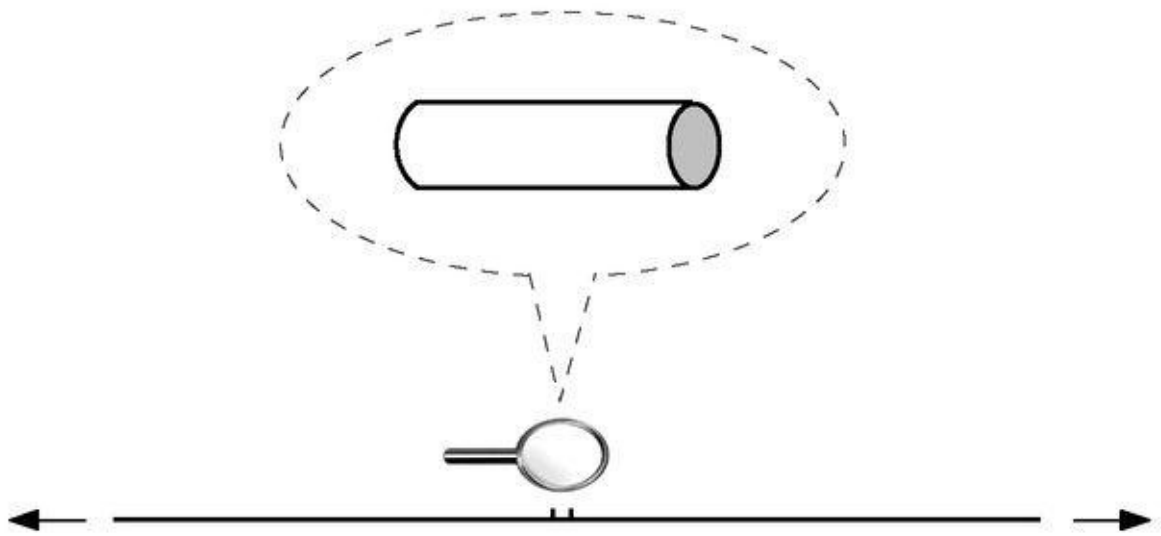
I may get in trouble for saying this, but I believe that only a mathematician would have been bold enough to think that higher-dimensional space would afford us special insight into phenomena that we've so far only managed to observe in a lower-dimensional setting. I say that because mathematicians deal with extra dimensions all the time. We're so comfortable with that notion, we don't give it a moment's thought. We could probably manipulate extra dimensions in our sleep without interfering with the REM phase.

Even if I think that only a mathematician would have made such a leap, in this case, remarkably, it was a mathematician building on the work of a physicist, Einstein. (And another physicist, Oskar Klein, whom we'll be discussing momentarily, soon built on that mathematician's work.) That's why I like to position myself at the interface between these two fields, math and physics, where a lot of interesting cross-pollination occurs. I've hovered around that fertile zone since the 1970s and have managed to get wind of many intriguing developments as a result.

But returning to Kaluza's provocative idea, people at the time were puzzled by a question that is equally valid today. And it's one that Kaluza undoubtedly grappled with as well: If there really is a fifth dimension—an entirely new direction to move at every point in our familiar four-dimensional world—how come nobody has seen it?

The obvious explanation is that this dimension is awfully small. But where would it be? One way to get a sense of that is to imagine our four-dimensional universe as a single line that extends endlessly in both directions. The idea here is that the three spatial dimensions are either extremely big or infinitely large. We'll also assume that time, too, can be mapped onto an infinite line—an assumption that may be questionable. At any rate, each point  $w$  on what we've thought of as a line actually represents a particular point  $(x, y, z, t)$  in four-dimensional spacetime.

In geometry, lines are normally just length, having no breadth whatsoever. But we're going to allow for the possibility that this line, when looked at with an exceedingly powerful magnifying glass, actually has some thickness. When seen in this light, our line is not really a line at all but rather an extremely slender cylinder or "garden hose," to choose the standard metaphor. Now, if we slice our hose at each point  $w$ , the cross-section of that cut will be a tiny circle, which, as we've said, is a one-dimensional curve. The circle thus represents the extra, fifth dimension that is "attached," in a sense, to every single point in four-dimensional spacetime.



1.5—Let’s picture our infinite, four-dimensional spacetime as a line that extends endlessly in both directions. A line, by definition, has no thickness. But if we were to look at that line with a magnifying glass, as suggested in the Kaluza-Klein approach, we might discover that the line has some thickness after all—that it is, in fact, harboring an extra dimension whose size is set by the diameter of the circle hidden within.

A dimension with that characteristic—being curled up in a tiny circle—is technically referred to as being compact. The word *compact* has a fairly intuitive meaning: Physicists sometimes say that a compact object or space is something you could fit into the trunk of your car. But there’s a more precise meaning as well: If you travel in one direction long enough, it is possible to return to the same spot. Kaluza’s five-dimensional spacetime includes both extended (infinite) and compact (finite) dimensions.

But if that picture were correct, why wouldn’t we notice ourselves moving around in circles in the fifth dimension? The answer to that question came in 1926 from Oskar Klein, the Swedish physicist who carried Kaluza’s idea a step further. Drawing on quantum theory, Klein actually calculated the size of the compact dimension, arriving at a number that was tiny indeed—close to the so-called Planck length, which is about as small as you can get—around  $10^{-30}$  cm in circumference.<sup>6</sup> And that’s how a fifth dimension could exist, yet remain forever unobservable. There is no foreseeable means by which we could see this minuscule dimension; nor could we detect movements within it.

Kaluza-Klein theory, as the work is now known, was truly remarkable, showing the potential of extra dimensions to demystify the secrets of nature. After sitting on Kaluza’s original paper for more than two years, Einstein wrote back saying he liked the idea “enormously.”<sup>7</sup> In fact, he liked the idea enough to pursue Kaluza-Klein-inspired approaches (sometimes in collaboration with the physicist Peter Bergmann) off and on over the next twenty years.

But ultimately, Kaluza-Klein theory was discarded. In part this was because it predicted a particle that has never been shown to exist, and in part because attempts to use the theory to compute the ratio of an electron’s mass to its charge went badly awry. Furthermore, Kaluza and Klein—as well as

Einstein after them—were trying to unify only electromagnetism and gravity, as they didn't know about the weak and strong forces, which were not well understood until the latter half of the twentieth century. So their efforts to unify all the forces were doomed to failure because the deck they were playing with was still missing a couple of important cards. But perhaps the biggest reason that Kaluza-Klein theory was cast aside had to do with timing: It was introduced just as the quantum revolution was beginning to take hold.

Whereas Kaluza and Klein put geometry at the center of their physical model, quantum theory was not only an ungeometric approach, but also one that directly conflicts with conventional geometry (which is the subject of Chapter 14). In the wake of the upheaval that ensued as quantum theory swept over physics in the twentieth century, and the amazingly productive period that followed, it took almost fifty years for the idea of new dimensions to be taken seriously again.

General relativity, the geometry-based theory that encapsulates our current understanding of gravity, has also held up extraordinarily well since Einstein introduced it in 1915, passing every experimental test it has faced. And quantum theory beautifully describes three of the known forces: the electromagnetic, weak, and strong. Indeed, it is the most precise theory we have, and “probably the most accurately tested theory in the history of human thought,” as Harvard physicist Andre Strominger has claimed.<sup>8</sup> Predictions of the behavior of an electron in the presence of an electric field, for example, agree with measurements to ten decimal points.

Unfortunately, these two very robust theories are totally incompatible. If you try to mix general relativity with quantum mechanics, the combination can create a horrific mess. The trouble arises from the quantum world, where things are always moving or fluctuating: The smaller the scale, the bigger those fluctuations get. The result is that on the tiniest scales, the turbulent, ever-changing picture afforded by quantum mechanics is totally at odds with the smooth geometric picture of spacetime upon which the general theory of relativity rests.

Everything in quantum mechanics is based on probabilities, and when general relativity is thrown into the quantum model, calculations often lead to infinite probabilities. When infinities pop up as a matter of course, that's a tipoff that something is amiss in your calculations. It's hardly an ideal state of affairs when your two most successful theories—one describing large objects such as planets and galaxies, and the other describing tiny objects such as electrons and quarks—combine to give you gibberish. Keeping them separate is not a satisfactory solution, either, because there are places, such as black holes, where the very large and very small converge, and neither theory on its own can make sense of them. “There shouldn't be laws of physics,” Strominger maintains. “There should be just one law and it ought to be the nicest law around.”<sup>9</sup>

Such a sentiment—that the universe can and should be describable by a “unified field theory” that weaves all the forces of nature into a seamless whole—is both aesthetically appealing and tied to the notion that our universe started with an intensely hot Big Bang. At that time, all the forces would have been at the same unimaginably high energy level and would therefore act as if they were a single



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