



THEORY
AND
APPLICATION
OF
INFINITE
SERIES

Konrad Knopp

THEORY AND APPLICATION OF INFINITE SERIES

BY

DR. KONRAD KNOPP

LATE PROFESSOR OF MATHEMATICS AT THE
UNIVERSITY OF TÜBINCEN

Translated
from the Second German Edition
and revised
in accordance with the Fourth by
Miss R. C. H. Young, Ph.D., L.èsSc.

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From the preface to the first (German) edition.

There is no general agreement as to where an account of the theory of infinite series should begin, what its main outlines should be, or what it should include. On the one hand, the whole of higher analysis may be regarded as a field for the application of this theory, for all limiting processes — including differentiation and integration — are based on the investigation of infinite sequences of infinite series. On the other hand, in the strictest (and therefore narrowest) sense, the only matters that are in place in a textbook on infinite series are their definition, the manipulation of the symbols connected with them, and the theory of convergence.

In his “Vorlesungen über Zahlen- und Funktionenlehre”, Vol. 1, Part 2, *A. Pringsheim* has treated the subject with these limitations. There was no question of offering anything similar in the present book.

My aim was quite different: namely, to give a comprehensive account of all the investigations of higher analysis in which infinite series are the chief object of interest, the treatment to be as free from assumptions as possible and to start at the very beginning and lead on to the extensive frontiers of present-day research. To set all this forth in as interesting and intelligible a way as possible, but of course without in the least abandoning exactness, with the object of providing the student with a convenient introduction to the subject and of giving him an idea of its rich and fascinating variety — such was my vision.

The material grew in my hands, however, and resisted my efforts to put it into shape. In order to make a convenient and useful book, the field had to be restricted. But I was guided throughout by the experience I have gained in teaching — I have covered the whole of the ground several times in the general course of my work and in lectures at the universities of Berlin and Königsberg — and also by the aim of the book. *It was to give a thorough and reliable treatment which would be of assistance to the student attending lectures and which would at the same time be adapted for private study.*

The latter aim was particularly dear to me, and this accounts for the form in which I have presented the subject-matter. Since it is generally easier — especially for beginners — to prove a deduction in pure mathematics than to recognize the restrictions to which the train of reasoning on the subject, I have always dwelt on *theoretical difficulties*, and have tried to remove them by means of repeated illustrations; and although I have thereby deprived myself of a good deal of space for an important matter, I hope to win the gratitude of the student.

I considered that an introduction to the theory of real numbers was indispensable as a beginning in order that the first facts relating to convergence might have a firm foundation. To this introduction I have added a fairly extensive account of the theory of sequences, and, finally, the actual theory of infinite series. The latter is then constructed in two storeys, so to speak: a ground-floor, in which the classical part of the theory (up to about the stage of *Cauchy's Analyse algébrique*) is expounded, though with the help of very limited resources, and a superstructure, in which I have attempted to give an account of the later developments of the 19th century.

For the reasons mentioned above, I have had to omit many parts of the subject to which I would gladly have given a place for their own sake. Semi-convergent series, *Euler's* summation formula, detailed treatment of the Gamma-function, problems arising from the hyper-geometric series, the theory of double series, the newer work on power series, and, in particular, a more thorough development of the last chapter, that on divergent series — all these I was reluctantly obliged to set aside. On the other hand, I considered that it was essential to deal with sequences and series of complex terms. As the theory runs almost parallel with that for real variables, however, I have, from the beginning, formulated all the definitions and proved all the theorems concerned in such a way that they remain valid without alteration, whether the “arbitrary” numbers involved are real or complex.

These definitions and theorems are further distinguished by the sign °.

~~In choosing the examples — in this respect, however, I lay no claim to originality; on the~~
contrary, in collecting them I have made extensive use of the literature — I have taken pains to put
practical applications in the fore-front and to leave mere playing with theoretical niceties alone.
Hence there are e. g. a particularly large number of exercises on Chapter VIII and only very few on
Chapter IX. Unfortunately there was no room for solutions or even for hints for the solution of the
examples.

A list of the most important papers, comprehensive accounts, and textbooks on infinite series
given at the end of the book, immediately in front of the index.

Königsberg, September 1921.

From the preface to the second (German) edition.

The fact that a second edition was called for after such a remarkably short time could be taken to mean that the first had on the whole been on the right lines. Hence the general plan has not been altered, but it has been improved in the details of expression and demonstration on almost every page.

The last chapter, that dealing with *divergent series*, has been wholly rewritten, with important extensions, so that it now in some measure provides an introduction to the theory and gives an idea of modern work on the subject.

Königsberg, December 1923.

Preface to the third (German) edition.

The main difference between the third and second editions is that it has become possible to add a new chapter on Euler's summation formula and asymptotic expansions, which I had reluctantly omitted from the first two editions. This important chapter had meanwhile appeared in a similar form in the English translation published by *Blackie & Son Limited*, London and Glasgow, in 1928.

In addition, the whole of the book has again been carefully revised, and the proofs have been improved or simplified in accordance with the progress of mathematical knowledge or teaching experience. This applies especially to theorems 269 and 287.

Dr. *W. Schöbe* and Herr *P. Securius* have given me valuable assistance in correcting the proofs for which I thank them heartily.

Tubingen, March 1931.

Preface to the fourth (German) edition.

In view of present difficulties no large changes have been made for the fourth edition, but the book has again been revised and numerous details have been improved, discrepancies removed, and several proofs simplified. The references to the literature have been brought up to date.

Tübingen, July 1947.

Preface to the first English edition.

This translation of the second German edition has been very skilfully prepared by Miss *R. C. Young*, L. ès Sc. (Lausanne), Research Student, Girton College, Cambridge. The publishers, Messrs *Blackie and Son, Ltd.*, Glasgow, have carefully superintended the printing.

In addition, the publishers were kind enough to ask me to add a chapter on *Eider's summation formula and asymptotic expansions*. I agreed to do so all the more gladly because, as I mentioned in the original preface, it was only with great reluctance that I omitted this part of the subject in the German edition. This chapter has been translated by Miss *W. M. Deans*, B.Sc. (Aberdeen), M.A. (Cantab.), with equal skill.

I wish to take this opportunity of thanking the translators and the publishers for the trouble and care they have taken. If — as I hope — my book meets with a favourable reception and is found useful by English-speaking students of Mathematics, the credit will largely be theirs.

Tubingen, February 1928.

Konrad Knopp

Preface to the second English edition.

The second English edition has been produced to correspond to the fourth German edition (1947)

Although most of the changes are individually small, they have nonetheless involved considerable number of alterations, about half of the work having been re-set.

The translation has been carried out by Dr. *R. C. H. Young* who was responsible for the original work.

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Introduction.

The foundation on which the structure of higher analysis rests is the *theory of real numbers*. A strict treatment of the foundations of the differential and integral calculus and of related subjects must inevitably start from there; and the same is true even for e. g. the calculation of roots and logarithms. The theory of real numbers first creates the material on which Arithmetic and Analysis can subsequently build, and with which they deal almost exclusively.

The necessity for this has not always been realized. The great creators of the infinitesimal calculus — *Leibniz* and *Newton*¹ — and the no less famous men who developed it, of whom *Euler*² is the chief, were too intoxicated by the mighty stream of learning springing from the newly-discovered sources to feel obliged to criticize fundamentals. To them the results of the new methods were sufficient evidence for the security of their foundations. It was only when the stream began to ebb that critical analysis ventured to examine the fundamental conceptions. About the end of the 18th century such efforts became stronger and stronger, chiefly owing to the powerful influence of *Gauss*³. Nearly a century had to pass, however, before the most essential matters could be considered thoroughly cleared up.

Nowadays rigour in connection with the underlying number concept is the most important requirement in the treatment of any mathematical subject. Ever since the later decades of the past century the last word on the matter has been uttered, so to speak, — by *Weierstrass*⁴ in the sixties, and by *Cantor*⁵ and *Dedekind*⁶ in 1872. No lecture or treatise dealing with the fundamental parts of higher analysis can claim validity unless it takes the refined concept of the real number as its starting-point.

Hence the theory of real numbers has been stated so often and in so many different ways since that time that it might seem superfluous to give another very detailed exposition⁷: for in this book (at least in the later chapters) we wish to address ourselves only to those already acquainted with the elements of the differential and integral calculus. Yet it would scarcely suffice merely to point to the accounts given elsewhere. For a theory of infinite series, as will be sufficiently clear from later developments, would be up in the clouds throughout, if it were not firmly based on the system of real numbers, the only possible foundation. On account of this, and in order to leave not the slightest uncertainty as to the hypotheses on which we shall build, we shall discuss in the following pages those ideas and data from the theory of real numbers which we shall need further on. We have no intention, however, of constructing a statement of the theory compressed into smaller space but otherwise complete. We merely wish to make the main ideas, the most important questions, and the answers to them, as clear and prominent as possible. So far as the latter are concerned, our treatment throughout will certainly be detailed and without omissions; it is only in the cases of details of subsidiary importance, and of questions as to the completeness and uniqueness of the system of real numbers which lie outside the plan of this book, that we shall content ourselves with shorter indications.

¹ *Gottfried Wilhelm Leibniz*, born in Leipzig in 1646, died in Hanover in 1716. *Isaac Newton*, born at Woolsthorpe in 1643, died in London in 1727. Each discovered the foundations of the infinitesimal calculus independently of the other.

² *Leonhard Euler*, born in Basle in 1707, died in St. Petersburg in 1783.

³ *Karl Friedrich Gauss*, born at Brunswick in 1777, died at Göttingen in 1855.

⁴ *Karl Weierstrass*, born at Ostenfelde in 1815, died in Berlin in 1897. The first rigorous account of the theory of real numbers which *Weierstrass* had expounded in his lectures since 1860 was given by *G. Mittag-Leffler*, one of his pupils, in his essay: *Die Zahlentheorie*, *Einleitung zur Theorie der analytischen Funktionen*, *The Tôhoku Mathematical Journal*, Vol. 17, pp. 157—209. 1920.

⁵ *Georg Cantor*, born in St. Petersburg in 1845, died at Halle in 1918: cf. *Mathem. Annalen*, Vol. 5, p. 123. 1872.

⁶ *Richard Dedekind*, born at Brunswick in 1831, died there in 1916: cf. his book: *Stetigkeit und irrationale Zahlen*, Brunswick, 1872.

⁷ An account which is easy to follow and which includes all the essentials is given by *H. v. Mangoldt*, *Einführung in die höhere Mathematik*, Vol. I, 8th edition (by K. Knopp), Leipzig 1944. — The treatment of *G. Kowalewski*, *Grundzüge der Differential- und Integralrechnung*, Vol. I, 1903.

Integralrechnung, 6th edition, Leipzig 1929, is accurate and concise. — A rigorous construction of the system of real numbers, which goes into the minutest details, is to be found in *A. Loewy*, *Lehrbuch der Algebra*, Part I, Leipzig 1915, in *A. Pringsheim*, *Vorlesung über Zahlen- und Funktionenlehre*, Vol. I, Part I, 2nd edition, Leipzig 1923 (cf. also the review of the latter work by *H. Hahn*, *Göttinger Anzeigen* 1919, pp. 321—47), and in a book by *E. Landau* exclusively devoted to this purpose, *Grundlagen der Analysis* (Das Rechnen mit ganzen, rationalen, irrationalen, komplexen Zahlen), Leipzig 1930. A critical account of the whole problem is to be found in the article by *F. Bachmann*, *Aufbau des Zahlensystems*, in the *Enzyklopädie d. math. Wissensch.*, Vol. I, 2nd edition, Part I, article 3, Leipzig and Berlin 1938.

Real numbers and sequences.

Principles of the theory of real numbers.

§ 1. The system of rational numbers and its gaps.

What do we mean by saying that a particular number is “known” or “given” or may be “calculated”? What does one mean by saying that he *knows* the value of $\sqrt{2}$ or π , or that he can *calculate* $\sqrt{5}$? A question like this is easier to ask than to answer. Were I to say that $\sqrt{2} = 1.41421356237$, I should obviously be wrong, since, on multiplying out, 1.414×1.414 does *not* give 2. If I assert, with greater caution, that $\sqrt{2} = 1.4142135 = 1.4142135$ and so on, even that is no tenable answer, and indeed in the first instance it is entirely meaningless. The question is, after all, *how* we are to go on, and this, without further indication, we cannot tell. Nor is the position improved by carrying the decimal further, even to hundreds of places. In this sense it may well be said that no one has ever beheld the whole of $\sqrt{2}$, — not held it completely in his own hands, so to speak—whilst a statement that $\sqrt{9} = 3 = 3$ or that $35 \div 7 = 5$ has a finished and thoroughly satisfactory appearance. The position is no better as regards the number π , or a logarithm or sine or cosine from the tables. Yet we feel certain that $\sqrt{2}$ and π and $\log 5$ really *do have* quite definite values, and even that we actually know these values. But a clear notion of what these impressions exactly amount to or imply we do not as yet possess. Let us endeavour to form such an idea.

Having raised doubts as to the justification for such statements as “I know $\sqrt{2}$ ”, we must, to be consistent, proceed to examine how far one is justified even in asserting that he *knows* the number — $\frac{23}{7}$ or is *given* (for some specific calculation) the number $\frac{9}{4}$. Nay more, the significance of such statements as “I know the number 97” or “for such and such a calculation I am *given* $a = 2$ and $b = 3$ ” would require scrutiny. We should have to enquire into the whole significance or *concept of the natural numbers* 1, 2, 3, . . .

This last question, however, strikes us at once as distinctly transgressing the bounds of Mathematics and as belonging to an order of ideas quite apart from that which we propose to develop here.

No science rests entirely within itself: each borrows the strength of its ultimate foundations from strata above or below it, such as experience, or theory of knowledge, or logic, or metaphysics, . . . Every science must accept *something* as simply given, and on that it may proceed to build. In this sense neither mathematics nor any other science starts without assumptions. The only question which has to be settled by a criticism of the foundation and logical structure of any science is what shall be assumed as in this sense “given”; or better, what minimum of initial assumptions will suffice, to serve as a basis for the subsequent development of all the rest.

For the problem we are dealing with, that of constructing the system of real numbers, the preliminary investigations are tedious and troublesome, and have actually, it must be confessed, not yet reached any entirely satisfactory conclusion at all. A discussion adequate to the present position of the subject would consequently take us far beyond the limits of the work we are contemplating. Instead, therefore, of shouldering an obligation to assume as basis only a minimum of hypotheses, we propose to regard at once as known (or “given”, or “secured”) a group of data whose deducibility from a smaller body of assumptions is familiar to everyone — namely, the *system of rational numbers*, i. e. of numbers integral and fractional, positive and negative, including zero. Speaking broadly, it is a matter of common knowledge how this system may be constructed, if — as a smaller body of assumptions — only the ordered sequence of natural numbers 1, 2, 3, . . . , and their combinations be

addition and multiplication, are regarded as “given”. For everyone knows — and we merely indicate in passing — how fractional numbers arise from the need of inverting the process of multiplication — negative numbers and zero from that of inverting the process of addition¹.

The totality, or aggregate, of numbers thus obtained is called the *system* (or *set*) of *rational numbers*. Each of these can be completely and literally “given” or “written down” or “made known” with the help of at most two natural numbers, a dividing bar and possibly a minus sign. For brevity we represent them by small italic characters; a, b, \dots, x, y, \dots . The following are the essential properties of this system :

1. Rational numbers form an *ordered* aggregate; meaning that between any two, say a and b , one and only one of the three relations

$$a < b, \quad a = b, \quad a > b$$

necessarily holds²; and these relations of “order” between rational numbers are subject to a set of quite simple laws, which we assume known, the only essential ones for our purposes being the

Fundamental Laws of Order.

1. Invariably³ $a = a$.
2. $a = b$ always implies $b = a$.
3. $a = b, b = c$ implies $a = c$.
4. $a \leq b, b < c$, — or $a < b, b \leq c$, — implies⁴ $a < c$.

2. Any two rational numbers may be combined in four distinct ways, referred to respectively as the four processes (or basic operations) of Addition, Subtraction, Multiplication, and Division. These operations can always be carried out to one definite result, with the single exception of division by zero which is undefined and should be regarded as an entirely impossible or meaningless process; the four processes also obey a number of simple laws, the so-called *Fundamental Laws of Arithmetic*, and further rules deducible therefrom.

These too we shall regard as known, and state, concisely, those **Fundamental Laws or Axioms of Arithmetic** from which all the others may be inferred, by purely formal rules (i. e. by the laws of pure logic).

I. Addition. 1. *Every* pair of numbers a and b has invariably associated with it a third, c , called their *sum* and denoted by $a + b$.

2. $a = a', b = b'$ always imply $a + b = a' + b'$.
3. Invariably, $a + b = b + a$ (Commutative Law).
4. Invariably, $(a + b) + c = a + (b + c)$ (Associative Law).
5. $a < b$ always implies $a + c < b + c$ (Law of Monotony).

II. Subtraction.

To *every* pair of numbers a and b there corresponds a third number c , such that $a + c = b$.

III. Multiplication.

1. To *every* pair of numbers a and b there corresponds a third number c , called their *product* and denoted by $a b$.

2. $a = a', b = b'$ always implies $a b = a' b'$.
3. In all cases $ab = ba$ (Commutative Law).
4. In all cases $(ab) c = a (b c)$ (Associative Law).

5. In all cases $(a + b)c = ac + bc$ (Distributive Law).

6. $a < b$ implies, provided c is positive, $ac > bc$ (Law of Monotony).

IV. Division.

To every pair of numbers a and b of which the first is not 0 there corresponds a third number such that $a c = b$.

As already remarked, all the known rules of arithmetic, — and hence ultimately all mathematical results, — are deduced from these few laws, with the help of the laws of pure logic alone. Among these laws, one is distinguished by its primarily mathematical character, namely the

V. Law of Induction, which may be reckoned among the fundamental laws of arithmetic and normally stated as follows:

If a set \mathfrak{M} of natural numbers includes the number 1, and if, every time a certain natural number and all those less than n can be taken to belong to the aggregate, the number $(n + 1)$ may be inferred also to belong to it, then \mathfrak{M} includes *all* the natural numbers.

This law of induction itself follows quite easily from the following theorem, which appears even more obvious and is therefore normally called the fundamental law of the natural numbers:

Law of the Natural Numbers. In every set of natural numbers that is not “empty” there is always a number less than all the rest.

For if, according to the hypotheses of the Induction Law, we consider the set \mathfrak{M} of natural numbers not belonging to \mathfrak{M} , this set \mathfrak{M} must be “empty”, that is, \mathfrak{M} must contain all the natural numbers. For otherwise, by the law of the natural numbers, \mathfrak{M} would include a number less than all the rest. This least number would exceed 1, for it was assumed that 1 belongs to \mathfrak{M} ; hence it could be denoted by $n + 1$. Then n would belong to \mathfrak{M} , but $(n + 1)$ would not, which contradicts the hypothesis in the law of induction.⁵

In applications it is usually an advantage to be able to make statements not merely about the natural numbers but about *any* whole numbers. The laws then take the following forms, obviously equivalent to those above:

Law of Induction. If a statement involves a natural number n (e. g. “if $n \geq 10$, then $2^n > n^3$ ”, or the like) and if

a) this statement is correct for $n = p$,

and

b) its correctness for $n = p, p + 1, \dots, k$ (where k is any natural number $\geq p$) always implies its correctness for $n = k + 1$, then the statement is correct for every natural number $\geq p$.

Law of Integers. In every set of integers all $\geq p$ that is not “empty”, there is always a number less than all the rest.⁶

We will lastly mention a theorem susceptible, in the domain of rational numbers, of immediate proof, although it becomes axiomatic in character very soon after this domain is left; namely the

VI. Theorem of Eudoxus.

If a and b are any two *positive* rational numbers, then a natural number n always exists⁷ such that $nb > a$.

The four ways of combining two rational numbers give in every case as the result another rational number. In this sense the system of rational numbers forms a *closed aggregate* (*natiirlicher Rationalitcits bereich* or *number corpus*). This property of forming a closed system with respect to the four rules is obviously not possessed by the aggregate of all natural numbers, or of all positive and negative integers. These are, so to speak, too sparsely sown to meet all the demands which the four rules make upon them.

This closed aggregate of all rational numbers and the laws which hold in it, are then all that we regard as given, known, secured.

As that type of argument which makes use of *inequalities* and *absolute values* 3. may be a little unfamiliar to some, its most important rules may be set down here, briefly and without proof:

I. Inequalities. Here all follows from the laws of order and monotony. In particular

1. The statements in the laws of monotony are reversible; e. g. $a + c < b + c$ always implies $a < b$; and so does $a c < b c$, provided $c > 0$.

2. $a < b$, $c < d$ always implies $a + c < b + d$.

3. $a < b$, $c < d$ implies, provided b and c are positive, $a c < b d$.

4. $a < b$ always implies $-b < -a$, and also, provided a is positive, $\frac{1}{b} < \frac{1}{a}$.

Also these theorems, as well as the laws of order and monotony, hold (with appropriate modifications) when the signs " \leq ", " $>$ ", " \geq ", and " $+$ " are substituted for " $<$ ", provided we maintain the assumptions that c , b and a are positive, in 1, 3, and 4 respectively.

II. Absolute values. *Definition:* By $|a|$, the *absolute value* (or *modulus*) of a , is meant that one of the two numbers $+a$ and $-a$ which is positive, supposing $a \neq 0$; and the number 0, if $a = 0$. (Hence $|a| = 0$ and if $a \neq 0$, $|a| > 0$.)

The following theorems hold, amongst others:

1. $|a| = |-a|$.

2. $|ab| = |a| \cdot |b|$.

3. $\left|\frac{1}{a}\right| = \frac{1}{|a|}$; $\left|\frac{b}{a}\right| = \frac{|b|}{|a|}$, provided $a \neq 0$.

4. $|a + b| \leq |a| + |b|$; $|a + b| \geq |a| - |b|$, and indeed $|a + b| \geq ||a| - |b||$.

5. The two relations $|a| < r$ and $-r < a < r$ are exactly equivalent; similarly for $|x - a| < r$ and $-r < x < a + r$.

6. $|a - b|$ is the *distance* between the *points* a and b , with the representation of numbers on a straight line described immediately below.

Proof of the first relation in 4: $\pm a \leq |a|$, $\pm b \leq |b|$ so that by 3, I, 2, $\pm(a + b) \leq |a| + |b|$, and hence $|a + b| \leq |a| + |b|$.

We also assume it to be known how the relations of magnitude between rational numbers may be illustrated graphically by relations of positions between points on a straight line. On a straight line, *number-axis*, any two distinct points are marked, one O , the origin (0) and one U , the unit point (1). The point P which is to represent a number $a = \frac{p}{q}$ ($q > 0$, $p \leq 0$, both integers) is obtained by marking off on the axis, $|p|$ times in succession, beginning at O , the q^{th} part of the distance OU (immediately constructed by elementary geometry) either in the direction OU , if $p > 0$, or if p is negative, in the opposite direction. This point⁸ we call for brevity *the point* a , and the totality of points corresponding in this way to all rational numbers we shall refer to as *the rational points* of the axis. — The straight line is usually thought of as drawn from *left to right* and U chosen to the right of O . In this case, the words *positive* and *negative* obviously become equivalents of the phrases: *to the right of* O and *to the left of* O , respectively; and, more generally, $a < b$ signifies that a lies to the left of b , b to the right of a . This mode of expression may often assist us in illustrating abstract relations between numbers.

This completes the sketch of what we propose to take as the previously secured foundation of our subject. We shall now regard the description of these foundations as characterizing the *concept* of a *number*; in other words, we shall call any system of conceptually well-distinguished objects

(elements, symbols) a *number system*, and its elements *numbers*, if—to put it quite briefly for the moment—we can operate with them in essentially the same ways as we do with rational numbers.

We proceed to give this somewhat inaccurate statement a precise formulation.

We consider a *system* S of well-distinguished objects, which we denote by a, β, \dots . S will be called a *number system* and its elements a, β, \dots will be called *numbers* if, besides being capable of definition exclusively by means of rational numbers (i. e. ultimately by means of natural numbers alone)⁹, these symbols a, β, \dots satisfy the following four conditions:

1. Between any two elements α and β of S one and only one of the three relations¹⁰

$$\alpha < \beta, \quad \alpha = \beta, \quad \alpha > \beta$$

necessarily holds (this is expressed briefly by saying that S is an *ordered system*) and these *relations of order* between the elements of S are subject to the same fundamental laws as their analogues in the system of rational numbers¹¹.

2. Four distinct methods of combining any two elements of S are defined, called Addition, Subtraction, Multiplication and Division. With a single exception, to be mentioned immediately (3), these processes can always be carried out to one definite result, and obey the same Fundamental Laws 2, I—IV, as their analogues in the system of the rational numbers¹². (The “zero” of the system, which must be known in order that the elements can be divided into positive and negative, is to be defined and explained in footnote 14 below.)

3. With every rational number we can associate an element of S (and all others “equal” to it) in such a manner that, if a and b denote rational numbers, α, β their associates from S :

a) the relation 1. holding between α and β is of the same form as that holding between a and b .

b) the element resulting from a combination of α and β (i. e. $\alpha + \beta, \alpha - \beta, \alpha \cdot \beta$, or $\alpha \div \beta$) has for its associated rational number the result of the similar combination of a and b (i. e. $a + b, a - b, a \cdot b$, or $a \div b$ respectively).

[This is also expressed, more shortly, by saying that the system S contains a sub-system *similar* and *isomorphous* to the system of rational numbers. Such a sub-system is in fact constituted by those elements of S which we have associated with rational numbers¹³.]

In such a correspondence, an element of S associated with the rational number zero, and all elements equal to it, may be shortly referred to as the “zero” of the system of elements. The exception mentioned in 2. then relates to division by zero¹⁴.

4. For any two elements α and β of S both standing in the relation “ $>$ ” to the “zero” of the system there exists a natural number n for which $n\beta > \alpha$. Here $n\beta$ denotes the sum $\beta + \beta + \dots + \beta$ containing the element β n times. (*Postulate of Eudoxus*; cf. 2, VI.)

To this abstract characterisation of the concept of number we will append the following remark¹⁵: If the system S contains no other elements than those corresponding to rational numbers as specified in 3, then our system does not differ in any essential feature from the system of rational numbers, but only in the (purely external) *designation* of the elements by symbols, or in the (purely practical) *interpretation* which we give to these symbols; differences almost as irrelevant, at bottom, as those which occur when we write figures at one time in Arabic characters, at another, in Roman or Chinese, or take them to denote now temperature, now velocity or electric charge. Disregarding the external characteristics of notation and practical interpretation, we should thus be perfectly justified in considering the system S as *identical* with the system of rational numbers and in this sense we may put $a = \alpha, b = \beta, \dots$.

If, however, the system S contains other elements besides the above mentioned, then we shall say that S *includes* the system of rational numbers, and is an *extension* of it. Whether a system of the

more comprehensive kind exists at all, remains for the moment an open question; but an example will come before our notice presently in the system of real numbers¹⁶.

Having thus agreed as to the amount of preliminary assumption we require, we may now drop a further argument on the subject, and again raise the question: *What do we mean by saying that we know the value of the number $\sqrt{2}$ or π ?*

It must in the first instance be termed altogether paradoxical that a number having its square equal to 2 does not exist in the system so far constructed¹⁷, — or, in geometrical language, that the point A of the number-axis, whose distance from O equals the diagonal of the square of side O , coincides with none of the “rational points”. For the rational numbers are *dense*, i. e. between any two of them (which are distinct) we can point out as many more as we please (since, if $a < b$, the n rational

numbers given by $a + v \frac{b-a}{n+1}$, for $v = 1, 2, \dots, n$, evidently all lie between a and b and are distinct from these and from one another); but they are not, as we might say, dense enough to symbolise all conceivable points. Rather, as the aggregate of all integers proved too scanty to meet the requirements of the four processes of arithmetic, so also the aggregate of all rational numbers contains too many gaps¹⁸ to satisfy the more exacting demands of root extraction. One feels, nevertheless, that a perfectly definite numerical value belongs to the point A and therefore to the symbol $\sqrt{2}$. What are the tangible facts which underlie this feeling?

Obviously, in the first instance, this: We do, it is true, know perfectly well that the values 1.4 or 1.41 or 1.414 etc. for $\sqrt{2}$ are inaccurate, in fact that these (rational) numbers have squares < 2 , i. e. are too small. But we also know that the values 1.5 or 1.42 or 1.415 etc. are in the same sense too large, that the value which we are attempting to reach would have therefore to lie between the corresponding too large and too small values. We thus reach the definite conviction that the value of $\sqrt{2}$ is within our grasp, although the given values are all incorrect. The root of this conviction can only lie in the fact that we have at our command a *process*, by which the above values may be continued *as far as we please*; we can, that is, form pairs of decimal fractions, with 1, 2, 3, . . . places of decimals, one of each pair being too large, and the other too small, and the two differing only by one unit in the last decimal place, i. e. by $\frac{1}{10^n}$, if n is the number of decimal places. As this difference may be made *as small as we please*, by sufficiently increasing the number n of given decimal places, we are taught through the above process to enclose the value which we are in search of between two numbers as near as we please to one another. By a metaphor, somewhat bold at the present stage, we say that through this process $\sqrt{2}$ itself is “given”, — in virtue of it, $\sqrt{2}$ is “known”, — by it, $\sqrt{2}$ may be “calculated”, and so on.

We have precisely the same situation with regard to any other value which cannot actually be denoted by a rational number, as for instance π , $\log 2$, $\sin 10^\circ$ etc. If we say, these numbers are *known*, nothing more is implied than that we know some process (in most cases an *extremely* laborious one) by which, as detailed in the case of $\sqrt{2}$, the desired value may be imprisoned, hemmed in, within narrower and narrower space between rational numbers, — and this space ultimately narrowed down as much as we please.

For the purpose of a somewhat more general and more accurate statement of these matters, we insert a discussion of sequences of rational numbers, provisional in character, but nevertheless of fundamental importance for all that comes after.

§ 2. Sequences of rational numbers¹.

In the process indicated above for calculating $\sqrt{2}$, successive well-defined rational numbers were constructed; their expression in decimal form was material in the description; from this form we now

propose to free it, and start with the following

~~Definition.~~ *If, by means of any suitable process of construction, we can form successively a first, a second, a third, . . . (rational) number and if to every positive integer n one and only one well defined (rational) number x_n thus corresponds, then the numbers*

$$x_1, x_2, x_3, \dots, x_n, \dots$$

(in this order, corresponding to the natural order of the integers 1, 2, 3, . . . n , . . .) are said to form a sequence. We denote it for brevity by (x_n) or (x_1, x_2, \dots) .

Examples.

1. $x_n = \frac{1}{n}$; i. e. the sequence $(\frac{1}{n})$, or $1, \frac{1}{2}, \frac{1}{3}, \dots, \frac{1}{n}, \dots$,
2. $x_n = 2^n$; i. e. the sequence 2, 4, 8, 16, . . .
3. $x_n = a^n$; i. e. the sequence a, a^2, a^3, \dots , where a is a given number.
4. $x_n = x_n = \frac{1}{2} \{1 - (-1)^n\}$; i. e. the sequence 1, 0, 1, 0, 1, 0, . . .
5. $x_n =$ the decimal fraction for $\sqrt{2}$, terminated at the n^{th} digit.
6. $x_n = \frac{(-1)^{n-1}}{n}$ i. e. the sequence $1, -1, -\frac{1}{2}, +\frac{1}{3}, -\frac{1}{4}, \dots$
7. Let $x_1 = 1, x_2 = 1, x_3 = x_1 + x_2 = 2$ and, generally, for $n \geq 3$, let $x_n = x_{n-1} + x_{n-2}$. obtain the sequence 1, 1, 2, 3, 5, 8, 13, 21, . . . , usually called *Fibonacci's* sequence.
8. $1, 2, \frac{1}{2}, -2, -\frac{1}{2}, 3, \frac{1}{3}, -3, -\frac{1}{3}, \dots$
9. $2, \frac{3}{2}, \frac{4}{3}, \frac{5}{4}, \dots, \frac{n+1}{n}, \dots$
10. $0, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \dots, \frac{n-1}{n}, \dots$
11. $x_n =$ the n^{th} prime number²; i. e. the sequence 2, 3, 5, 7, 11, 13, . . .
12. The sequence $1, \frac{3}{2}, \frac{11}{6}, \frac{25}{12}, \frac{137}{60}, \dots$, in which $x_n = \left(1 + \frac{1}{2} + \dots + \frac{1}{n}\right)$.

Remarks.

1. The law of formation may be quite arbitrary; it need not, in particular, be embodied in an explicit formula enabling us to obtain x_n , for a given n , by direct calculation. In examples 6, 5, 7 and 11, clearly no such formula can be immediately written down. If the terms of the sequence are individually given, neither the law of formation (cf. 6, 5 and 12) nor any other kind of regularity (cf. 11) among the successive numbers is necessarily apparent.

2. It is sometimes advantageous to start the sequence with a "0th" term x_0 , or even with a (-1)th or (-2)th term, x_{-1}, x_{-2} . Occasionally, it pays better to start indexing with 2 or 3. The only essential

is that there should be an integer $1, \frac{3}{2}, \frac{11}{6}, \frac{25}{12}, \frac{137}{60}, \dots$, such that x_n is defined for every $n \geq m$. The term x_m is then called the *initial term* of the sequence. We will however, even then, continue to designate as the n^{th} term that which bears the index n . In § 6, 2, 3 and 4, for instance, we can without further difficulties take a 0th term or even (-1)th or (-2)th to head the sequence. The "first term" of a sequence is then not necessarily the term with which the sequence begins. The notation will be preferably (x_0, x_1, \dots) or (x_{-1}, x_0, \dots) , etc., as the case may be, unless it is either quite clear

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