THEORY AND APPLICATION OF INFINITE SERIES

Konrad Knopp

THEORY AND APPLICATION OF INFINITE SERIES

BY

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Translated
from the Second German Edition
and revised
in accordance with the Fourth by
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From the preface to the first (German) edition.

There is no general agreement as to where an account of the theory of infinite series shou begin, what its main outlines should be, or what it should include. On the one hand, the whole higher analysis may be regarded as a field for the application of this theory, for all limiting processe — including differentiation and integration — are based on the investigation of infinite sequences of infinite series. On the other hand, in the strictest (and therefore narrowest) sense, the only matte that are in place in a textbook on infinite series are their definition, the manipulation of the symbolis connected with them, and the theory of convergence.

In his "Vorlesungen über Zahlen- und Funktionenlehre", Vol. 1, Part 2, *A. Pringsheim* has treate the subject with these limitations. There was no question of offering anything similar in the prese book.

My aim was quite different: namely, to give a comprehensive account of all the investigations higher analysis in which infinite series are the chief object of interest, the treatment to be as free fro assumptions as possible and to start at the very beginning and lead on to the extensive frontiers present-day research. To set all this forth in as interesting and intelligible a way as possible, but course without in the least abandoning exactness, with the object of providing the student with convenient introduction to the subject and of giving him an idea of its rich and fascinating variety such was my vision.

The material grew in my hands, however, and resisted my efforts to put it into shape. In order make a convenient and useful book, the field had to be restricted. But I was guided throughout by t experience I have gained in teaching — I have covered the whole of the ground several times in t general course of my work and in lectures at the universities of Berlin and Königsberg — and also b the aim of the book. *It was to give a thorough and reliable treatment which would be of assistance the student attending lectures and which would at the same time be adapted for private study*.

The latter aim was particularly dear to me, and this accounts for the form in which I hav presented the subject-matter. Since it is generally easier — especially for beginners — to prove deduction in pure mathematics than to recognize the restrictions to which the train of reasoning subject, I have always dwelt on *theoretical difficulties,* and have tried to remove them by means repeated illustrations; and although I have thereby deprived myself of a good deal of space f important matter, I hope to win the gratitude of the student.

I considered that an introduction to the theory of real numbers was indispensable as a beginnin in order that the first facts relating to convergence might have a firm foundation. To this introducti I have added a fairly extensive account of the theory of sequences, and, finally, the actual theory infinite series. The latter is then constructed in two storeys, so to speak: a ground-floor, in which t classical part of the theory (up to about the stage of *Cauchy's* Analyse algébrique) is expounde though with the help of very limited resources, and a superstructure, in which I have attempted to gi an account of the later developments of the 19^{th} century.

For the reasons mentioned above, I have had to omit many parts of the subject to which I wou gladly have given a place for their own sake. Semi-convergent series, *Euler's* summation formula, detailed treatment of the Gamma-function, problems arising from the hyper-geometric series, t theory of double series, the newer work on power series, and, in particular, a more thoroug development of the last chapter, that on divergent series — all these I was reluctantly obliged to s aside. On the other hand, I considered that it was essential to deal with sequences and series complex terms. As the theory runs almost parallel with that for real variables, however, I have, fro the beginning, formulated all the definitions and proved all the theorems concerned in such a way th they remain valid without alteration, whether the "arbitrary" numbers involved are real or comple

These definitions and theorems are further distinguished by the sign °.

~~In choosing the examples — in this respect, however, I lay no claim to originality;~~ on th contrary, in collecting them I have made extensive use of the literature — I have taken pains to p practical applications in the fore-front and to leave mere playing with theoretical niceties alon Hence there are e. g. a particularly large number of exercises on Chapter VIII and only very few o Chapter IX. Unfortunately there was no room for solutions or even for hints for the solution of th examples.

A list of the most important papers, comprehensive accounts, and textbooks on infinite series given at the end of the book, immediately in front of the index.

Königsberg, September 1921.

From the preface to the second (German) edition.

The fact that a second edition was called for after such a remarkably short time could be taken to mean that the first had on the whole been on the right lines. Hence the general plan has not been altered, but it has been improved in the details of expression and demonstration on almost every page.

The last chapter, that dealing with *divergent series,* has been wholly rewritten, with important extensions, so that it now in some measure provides an introduction to the theory and gives an idea of modern work on the subject.

Königsberg, December 1923.

Preface to the third (German) edition.

The main difference between the third and second editions is that it has become possible to add new chapter on Euler's summation formula and asymptotic expansions, which I had reluctant omitted from the first two editions. This important chapter had meanwhile appeared in a similar for in the English translation published by *Blackie & Son Limited,* London and Glasgow, in 1928.

In addition, the whole of the book has again been carefully revised, and the proofs have bee improved or simplified in accordance with the progress of mathematical knowledge or teachi experience. This applies especially to theorems 269 and 287.

Dr. *W. Schöbe* and Herr *P. Securius* have given me valuable assistance in correcting the proof for which I thank them heartily.

Tubingen, March 1931.

Preface to the fourth (German) edition.

In view of present difficulties no large changes have been made for the fourth edition, but th book has again been revised and numerous details have been improved, discrepancies removed, an several proofs simplified. The references to the literature have been brought up to date.

Tübingen, July 1947.

Preface to the first English edition.

This translation of the second German edition has been very skilfully prepared by Miss *R. C. I* *Young,* L. ès Sc. (Lausanne), Research Student, Girton College, Cambridge. The publishers, Messr *Blackie and Son, Ltd.,* Glasgow, have carefully superintended the printing.

In addition, the publishers were kind enough to ask me to add a chapter on *Eider's summatic formula and asymptotic expansions*. I agreed to do so all the more gladly because, as I mentioned the original preface, it was only with great reluctance that I omitted this part of the subject in th German edition. This chapter has been translated by Miss *W. M. Deans,* B.Sc. (Aberdeen), M.A (Cantab.), with equal skill.

I wish to take this opportunity of thanking the translators and the publishers for the trouble ar care they have taken. If — as I hope —my book meets with a favourable reception and is found usef by English-speaking students of Mathematics, the credit will largely be theirs.

Tubingen, February 1928.

Konrad Knop

Preface to the second English edition.

The second English edition has been produced to correspond to the fourth German edition (1947

Although most of the changes are individually small, they have nonetheless involved considerable number of alterations, about half of the work having been re-set.

The translation has been carried out by Dr. *R. C. H. Young* who was responsible for the origin work.

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Introduction.

The foundation on which the structure of higher analysis rests is the *theory of real numbers*. Ar strict treatment of the foundations of the differential and integral calculus and of related subjects mu inevitably start from there; and the same is true even for e. g. the calculation of roots and logarithm The theory of real numbers first creates the material on which Arithmetic and Analysis ca subsequently build, and with which they deal almost exclusively.

The necessity for this has not always been realized. The great creators of the infinitesim calculus — *Leibniz* and *Newton*[1] — and the no less famous men who developed it, of whom *Euler*[2] the chief, were too intoxicated by the mighty stream of learning springing from the newly-discovere sources to feel obliged to criticize fundamentals. To them the results of the new methods we sufficient evidence for the security of their foundations. It was only when the stream began to ebb th critical analysis ventured to examine the fundamental conceptions. About the end of the 18th centu such efforts became stronger and stronger, chiefly owing to the powerful influence of *Gauss*[3]. Near a century had to pass, however, before the most essential matters could be considered thorough cleared up.

Nowadays rigour in connection with the underlying number concept is the most importa requirement in the treatment of any mathematical subject. Ever since the later decades of the pa century the last word on the matter has been uttered, so to speak, — by *Weierstrass*[4] in the sixties, ar by *Cantor*[5] and *Dedekind*[6] in 1872. No lecture or treatise dealing with the fundamental parts of high analysis can claim validity unless it takes the refined concept of the real number as its starting-point.

Hence the theory of real numbers has been stated so often and in so many different ways sin that time that it might seem superfluous to give another very detailed exposition[7]: for in this book (least in the later chapters) we wish to address ourselves only to those already acquainted with th elements of the differential and integral calculus. Yet it would scarcely suffice merely to point t accounts given elsewhere. For a theory of infinite series, as will be sufficiently clear from lat developments, would be up in the clouds throughout, if it were not firmly based on the system of re numbers, the only possible foundation. On account of this, and in order to leave not the slighte uncertainty as to the hypotheses on which we shall build, we shall discuss in the following pages tho ideas and data from the theory of real numbers which we shall need further on. We have no intentio however, of constructing a statement of the theory compressed into smaller space but otherwi complete. We merely wish to make the main ideas, the most important questions, and the answers t them, as clear and prominent as possible. So far as the latter are concerned, our treatment througho will certainly be detailed and without omissions; it is only in the cases of details of subsidia importance, and of questions as to the completeness and uniqueness of the system of real numbe which lie outside the plan of this book, that we shall content ourselves with shorter indications.

[1] *Gottfried Wilhelm Leibniz,* born in Leipzig in 1646, died in Hanover in 1716. *Isaac Newton,* born at Woolsthorpe in 164 died in London in 1727. Each discovered the foundations of the infinitesimal calculus independently of the other.

[2] *Leonhard Euler,* born in Basle in 1707, died in St. Petersburg in 1783.

[3] *Karl Friedrich Gauss,* born at Brunswick in 1777, died at Göttingen in 1855.

[4] *Karl Weierstrass,* born at Ostenfelde in 1815, died in Berlin in 1897. The first rigorous account of the theory of real numbe which *Weierstrass* had expounded in his lectures since 1860 was given by *G. Mittag-Leffler,* one of his pupils, in his essay: Die Za Einleitung zur Theorie der analytischen Funktionen, The Tôhoku Mathematical Journal, Vol. 17, pp. 157—209. 1920.

[5] *Georg Cantor,* born in St. Petersburg in 1845, died at Halle in 1918: cf. Mathem. Annalen, Vol. 5, p. 123. 1872.

[6] *Richard Dedekind,* born at Brunswick in 1831, died there in 1916: cf. his book: Stetigkeit und irrationale Zahlen, Brunswi 1872.

[7] An account which is easy to follow and which includes all the essentials is given by *H. v. Mangoldt,* Einführung in die höhe Mathematik, Vol. I, 8th edition (by K. Knopp), Leipzig 1944. — The treatment of *G. Kowalewski,* Grundzüge der Differential- u

Integralrechnung, 6th edition, Leipzig 1929, is accurate and concise. — A rigorous construction of the system of real numbers, whi goes ~~into the minutest details, is to be found in *A. Loewy,* Lehrbuch der Algebra, Part I, Leipzig 1915, in *A. Pringsheim,*~~ Vorlesung über Zahlen- und Funktionenlehre, Vol. I, Part I, 2nd edition, Leipzig 1923 (cf. also the review of the latter work by *H. Hahn,* Gö gel. Anzeigen 1919, pp. 321—47), and in a book by *E. Landau* exclusively devoted to this purpose, Grundlagen der Analysis (D Rechnen mit ganzen, rationalen, irrationalen, komplexen Zahlen), Leipzig 1930. A critical account of the whole problem is to found in the article by *F. Bachmann,* Aufbau des Zahlensystems, in the Enzyklopädie d. math. Wissensch., Vol. I, 2nd edition, Part article 3, Leipzig and Berlin 1938.

Real numbers and sequences.

Chapter I.

Principles of the theory of real numbers.

§ 1. The system of rational numbers and its gaps.

What do we mean by saying that a particular number is “known” or “given” or may b “calculated”? What does one mean by saying that he *knows* the value of $\sqrt{2}$ or π, or that he *ca calculate* $\sqrt{5}$? A question like this is easier to ask than to answer. Were I to say that $\sqrt{2} = 1{\cdot}414 =$ · 414, I should obviously be wrong, since, on multiplying out, 1 · 414 × 1 · 414 does *not* give 2. If assert, with greater caution, that $\sqrt{2} = 1{\cdot}4142135$ = 1 · 4142135 *and so on,* even that is no tenab answer, and indeed in the first instance it is entirely meaningless. The question is, after all, *how* we a to go on, and this, without further indication, we cannot tell. Nor is the position improved by carryir the decimal further, even to hundreds of places. In this sense it may well be said that no one has ev beheld the whole of $\sqrt{2}$, — not held it completely in his own hands, so to speak—whilst a stateme that $\sqrt{9} = 3$ = 3 or that 35 ÷ 7 = 5 has a finished and thoroughly satisfactory appearance. The positic is no better as regards the number π, or a logarithm or sine or cosine from the tables. Yet we fe certain that $\sqrt{2}$ and *π* and log 5 really *do have* quite definite values, and even that we actually kno these values. But a clear notion of what these impressions exactly amount to or imply we do not as y possess. Let us endeavour to form such an idea.

Having raised doubts as to the justification for such statements as “I know $\sqrt{2}$”, we must, to b consistent, proceed to examine how far one is justified even in asserting that he *knows* the number – $\frac{23}{7}$ or is *given* (for some specific calculation) the number $\frac{9}{4}$. Nay more, the significance of suc statements as “I know the number 97” or “for such and such a calculation I am *given a* = 2 and *b* = would require scrutiny. We should have to enquire into the whole significance or *concept of th natural numbers* 1, 2, 3, . . .

This last question, however, strikes us at once as distinctly transgressing the bounds o Mathematics and as belonging to an order of ideas quite apart from that which we propose to develo here.

No science rests entirely within itself: each borrows the strength of its ultimate foundations fro strata above or below it, such as experience, or theory of knowledge, or logic, or metaphysics, . Every science must accept *something* as simply given, and on that it may proceed to build. In th sense neither mathematics nor any other science starts without assumptions. The only question whic has to be settled by a criticism of the foundation and logical structure of any science is what shall b assumed as in this sense “given”; or better, what minimum of initial assumptions will suffice, to serv as a basis for the subsequent development of all the rest.

For the problem we are dealing with, that of constructing the system of real numbers, the preliminary investigations are tedious and troublesome, and have actually, it must be confessed, n yet reached any entirely satisfactory conclusion at all. A discussion adequate to the present position o the subject would consequently take us far beyond the limits of the work we are contemplatin Instead, therefore, of shouldering an obligation to assume as basis only a minimum of hypotheses, w propose to regard at once as known (or “given”, or “secured”) a group of data whose deducibility fro a smaller body of assumptions is familiar to everyone — namely, the *system of rational numbers,* i. of numbers integral and fractional, positive and negative, including zero. Speaking broadly, it is matter of common knowledge how this system may be constructed, if — as a smaller body assumptions — only the ordered sequence of natural numbers 1, 2, 3, . . . , and their combinations b addition and multiplication, are regarded as "given". For everyone knows — and we merely indicate in passing ~~— how fractional numbers arise from the need of inverting the process of multiplicatio~~ — negative numbers and zero from that of inverting the process of addition[1].

The totality, or aggregate, of numbers thus obtained is *called the system* (or *set) of ration* *numbers.* Each of these can be completely and literally *"given"* or *"written down"* or *"made know* with the help of at most two natural numbers, a dividing bar and possibly a minus sign. For brevit we represent them by small italic characters; *a, b,* . . . , *x*, *y*, . . . The following are the essenti properties of this system :

1. Rational numbers form an *ordered* aggregate; meaning that between any two, say *a* and *b,* o and only one of the three relations

$$a < b. \quad a = b, \quad a > b$$

necessarily holds[2]; and these relations of "order" between rational numbers are subject to a set quite simple laws, which we assume known, the only essential ones for our purposes being the

Fundamental Laws of Order.

1. Invariably[3] $a = a$.
2. $a = b$ always implies $b = a$.
3. $a = b$, $b = c$ implies $a = c$.
4. $a \leqq b$, $b < c$, — or $a < b$, $b \leqq c$, — implies[4] $a < c$.

2. Any two rational numbers may be combined in four distinct ways, referred to respectively the four processes (or basic operations) of Addition, Subtraction, Multiplication, and Division. The operations can always be carried out to one definite result, with the single exception of division by which is undefined and should be regarded as an entirely impossible or meaningless process; the fo processes also obey a number of simple laws, the so-called *Fundamental Laws of Arithmetic,* a further rules deducible therefrom.

These too we shall regard as known, and state, concisely, those **Fundamental Laws or Axion of Arithmetic** from which all the others may 2. be inferred, by purely formal rules (i. e. by the laws pure logic).

I. Addition. 1. *Every* pair of numbers *a* and *b* has invariably associated with it a third, *c,* call their *sum* and denoted by $a + b$.

2. $a = a'$, $b = b'$ always imply $a + b = a' + b'$.
3. Invariably, $a + b = b + a$ (Commutative Law).
4. Invariably, $(a + b) + c = a + (b + c)$ (Associative Law).
5. $a < b$ always implies $a + c < b + c$ (Law of Monotony).

II. Subtraction.

To *every* pair of numbers *a* and *b* there corresponds a third number *c,* such that $a + c = b$.

III.Multiplication.

1. To *every* pair of numbers *a* and *b* there corresponds a third number *c,* called their *product* a denoted by $a\,b$.
2. $a = a'$, $b = b'$ always implies $a\,b = a'\,b'$.
3. In all cases $ab = ba$ (Commutative Law).
4. In all cases $(ab)\,c = a\,(b\,c)$ (Associative Law).

5. In all cases $(a + b)c = ac + bc$ (Distributive Law).

~~6. $a < b$ implies, *provided c is positive*, $ac > bc$ (Law of Monotony).~~

IV.Division.

To *every* pair of numbers a and b of which the first is not 0 there corresponds a third number such that $a\,c = b$.

As already remarked, all the known rules of arithmetic, — and hence ultimately all mathematic results, — are deduced from these few laws, with the help of the laws of pure logic alone. Amor these laws, one is distinguished by its primarily mathematical character, namely the

V.Law of Induction, which may be reckoned among the fundamental laws of arithmetic and normally stated as follows:

If a set $\mathfrak{M}$ of natural numbers includes the number 1, and if, every time a certain natural number and all those less than n can be taken to belong to the aggregate, the number $(n + 1)$ may be inferre also to belong to it, then $\mathfrak{M}$ includes *all* the natural numbers.

This law of induction itself follows quite easily from the following theorem, which appears eve more obvious and is therefore normally called the fundamental law of the natural numbers:

Law of the Natural Numbers. In every set of natural numbers that is not "empty" there always a number less than all the rest.

For if, according to the hypotheses of the Induction Law, we consider the set $\mathfrak{M}$ of natur numbers not belonging to $\mathfrak{M}$, this set $\mathfrak{M}$ must be "empty", that is, $\mathfrak{M}$ must contain all the natur numbers. For otherwise, by the law of the natural numbers, $\mathfrak{M}$ would include a number less than all tl rest. This least number would exceed 1, for it was assumed that 1 belongs to $\mathfrak{M}$; hence it could l denoted by $n + 1$. Then n would belong to $\mathfrak{M}$, but $(n + 1)$ would not, which contradicts the hypothes in the law of induction.[5]

In applications it is usually an advantage to be able to make statements not merely about tl natural numbers but about *any* whole numbers. The laws then take the following forms, obvious equivalent to those above:

Law of Induction. If a statement involves a natural number n (e. g. "if $n \geqq 10$, then $2^n > n^3$", or tl like) and if

a) this statement is correct for $n = p$,

and

b) its correctness for $n = p, p + 1, \dots, k$ (where k is any natural number $\geqq p$) always implies i correctness for $n = k + 1$, then the statement is correct for every natural number $\geqq p$.

Law of Integers. In every set of integers all $\geqq p$ that is not "empty", there is always a number le than all the rest.[6]

We will lastly mention a theorem susceptible, in the domain of rational numbers, of immedia proof, although it becomes axiomatic in character very soon after this domain is left; namely the

VI. Theorem of Eudoxus.

If a and b are any two *positive* rational numbers, then a natural number n always exists[7] such th $nb > a$.

The four ways of combining two rational numbers give in every case as the result another ration number. In this sense the system of rational numbers forms a *closed aggregate (natiirlich Rationalitcits bereich* or *number corpus)*. This property of forming a closed system with respect to tl four rules is obviously not possessed by the aggregate of all natural numbers, or of all positive ar negative integers. These are, so to speak, too sparsely sown to meet all the demands which the fo rules make upon them.

This closed aggregate of all rational numbers and the laws which hold in it, are then all that w regard as given, known, secured.

As that type of argument which makes use of *inequalities* and *absolute values* 3. may be a litt unfamiliar to some, its most important rules may be set down here, briefly and without proof:

I. Inequalities. Here all follows from the laws of order and monotony. In particular

1. The statements in the laws of monotony are reversible; e. g. $a + c < b + c$ always implies $< b$; and so does $a\,c < b\,c$, *provided* $c > 0$.
2. $a < b,\ c < d$ always implies $a + c < b + d$.
3. $a < b,\ c < d$ implies, provided b and c are positive, $a\,c < b\,d$.
4. $a < b$ always implies $-b < -a$, and also, provided a is positive, $\frac{1}{b} < \frac{1}{a}$.

Also these theorems, as well as the laws of order and monotony, hold (with appropria modifications) when the signs "$\leqq$", "$>$", "$\geqq$", and "+ " are substituted for "$<$", provided v maintain the assumptions that c, b and a are positive, in 1, 3, and 4 respectively.

II. Absolute values. *Definition:* By $|a|$, the*absolute value* (or *modulus)* of a, is meant that one the two numbers $+a$ and$-a$ which is positive, sup posing $a \neq 0$; and the number 0, if $a = 0$. (Hence $|$ $| = 0$ and if $a \neq 0$, $|a| > 0$.)

The following theorems hold, amongst others:

1. $|a| = |-a|$.
2. $|ab| = |a| \cdot |b|$.
3. $\left|\frac{1}{a}\right| = \frac{1}{|a|}$; $\qquad \left|\frac{b}{a}\right| = \frac{|b|}{|a|}$, provided $a \neq 0$.
4. $|a + b| \leqq |a| + |b|$; $|a + b| \geqq |a| - |b|$, and indeed $|a + b| \geqq \big||a| - |b|\big|$.
5. The two relations $|a| < r$ and $-r < a < r$ are exactly equivalent; similarly for $|x - a| < r$ and $-r < x < a + r$.
6. $|a - b|$ is the *distance* between the *points* a and b, with the represen tation of numbers on straight line described immediately below.

Proof of the first relation in 4: $\pm a \leqq |a|$, $\pm b$ so that by 3, I, 2, $\pm(a + b) \leqq |a| + b|$, and henc $a + b| \leqq a| + |b|$.

We also assume it to be known how the relations of magnitude between rational numbers may b illustrated graphically by relations of positions between points on a straight line. On a straight line *number-axis,* any two distinct points are marked, one *O,* the origin (0) and one *U,* the unit point (1 The point *P* which is to represent a number $a = \frac{p}{q}$ ($q > 0$, $p \lesseqgtr 0$, both integers) is obtained by markin off on the axis, $|p|$ times in succession, beginning at *O,* the q^{th} part of the distance *O U* (immediate constructed by elementary geometry) either in the direction *O U,* if $p > 0$, or if p is negative, in th opposite direction. This point[8] we call for brevity *the point a,* and the totality of points correspondin in this way to all rational numbers we shall refer to as *the rational points* of the axis. — The straig line is usually thought of as drawn from *left to right* and *U* chosen to the right of *O.* In this case, th words *positive* and *negative* obviously become equivalents of the phrases: *to the right of O* and *to th left of O,* respectively; and, more generally, $a < b$ signifies that a lies to the left of b, b to the right *a.* This mode of expression may often assist us in illustrating abstract relations between numbers.

This completes the sketch of what we propose to take as the previously secured foundation of o subject. We shall now regard the description of these foundations as characterizing the ***concept number;*** in other words, we shall call any system of conceptually well-distinguished objec (elements, symbols) a *number system,* and its elements ***numbers,*** if—to put it quite briefly for th ~~moment — we can operate with them in essentially the same ways as we do with rational numbers~~.

We proceed to give this somewhat inaccurate statement a precise formulation.

We consider a *system S* of well-distinguished objects, which we denote by a, β, ... *S* will b called a *number system* and its elements *a*, *β*, … will be called *numbers* if, besides being capable definition exclusively by means of rational numbers (i. e. ultimately by means of natural numbe alone)[9], these symbols a, *β*, . . . satisfy the following four conditions:

1. Between any two elements α and *β* of *S* one and only one of the three relations[10]

$$\alpha < \beta, \quad \alpha = \beta, \quad \alpha > \beta$$

necessarily holds (this is expressed briefly by saying that *S is an ordered system)* and these *relatio of order* between the elements of *S* are subject to the same fundamental laws 1 as their analogues the system of rational numbers[11].

2. Four distinct methods of combining any two elements of *S* are defined, called Additio Subtraction, Multiplication and Division. With a single exception, to be mentioned immediately (3 these processes can always be carried out to one definite result, and obey the same Fundamental Law 2, I—IV, as their analogues in the system of the rational numbers[12]. (The “zero” of the system, whic must be known in order that the elements can be divided into positive and negative, is to be defined explained in footnote 14 below.)

3. With every rational number we can associate an element of *S* (and all others “equal” to it) such a manner that, if *a* and *b* denote rational numbers, a, *β* their associates from *S:*

a) the relation 1. holding between α and *β* is of the same form as that holding between *a* and *b.*

b) the element resulting from a combination of a and . (i. e. α + β, α — β, α · β, or α ÷ β) has f its associated rational number the result of the similar combination of *a* and *b* (i. e. *a* + *b, a* — *b, a* · or *a* ÷ *b* respectively).

[This is also expressed, more shortly, by saying that the system *S* contains a sub-system ***similar*** and ***isowiorphous*** to the system of rational numbers. Such a sub-system is in fact constitute by those elements of *S* which we have associated with rational numbers[13].]

In such a correspondence, an element of *S* associated with the rational number zero, and a elements equal to it, may be shortly referred to as the “zero” of the system of elements. The exceptio mentioned in 2. then relates to division by zero[14].

4. For any two elements α and β of *S* both standing in the relation “>“ to the “zero” of the syster there exists a natural number *n* for which *n β* > α. Here *n β* denotes the sum *β* + *β* + . . . + *β* containi the element *β n* times. ***(Postulate of Eudoxus;*** cf. **2**, VI.)

To this abstract characterisation of the concept of number we will append the followin remark[15]: If the system *S* contains no other elements than those corresponding to rational numbers specified in 3, then our system does not differ in any essential feature from the system of ration numbers, but only in the (purely external) *designation* of the elements by symbols, or in the (purel practical) *interpretation* which we give to these symbols; differences almost as irrelevant, at bottor as those which occur when we write figures at one time in Arabic characters, at another, in Roman Chinese, or take them to denote now temperature, now velocity or electric charge. Disregardin external characteristics of notation and practical interpretation, we should thus be perfectly justified considering the system S as *identical* with the system of rational numbers and in this sense we ma put *a* = α, *b*=*β*, ...

If, however, the system *S* contains other elements besides the above mentioned, then we shall sa that *S includes* the system of rational numbers, and is an *extension* of it. Whether a system of th more comprehensive kind exists at all, remains for the moment an open question; but an example w come before our notice presently in the system of real numbers[16].

Having thus agreed as to the amount of preliminary assumption we require, we may now drop a argument on the subject, and again raise the question: *What do we mean by saying that we know t number $\sqrt{2}$ or π?*

It must in the first instance be termed altogether paradoxical that a number having its squa equal to 2 does not exist in the system so far constructed[17], — or, in geometrical language, that t point A of the number-axis, whose distance from O equals the diagonal of the square of side O coincides with none of the "rational points". For the rational numbers are *dense,* i. e. between any tw of them (which are distinct) we can point out as many more as we please (since, if $a < b$, the n ration numbers given by $a + \nu\frac{b-a}{n+1}$, for $\nu = 1, 2, \ldots, n$, evidently all lie between a and b and are distin from these and from one another); but they are not, as we might say, dense enough to symbolise a conceivable points. Rather, as the aggregate of all integers proved too scanty to meet the requiremen of the four processes of arithmetic, so also the aggregate of all rational numbers contains too ma gaps[18] to satisfy the more exacting demands of root extraction. One feels, nevertheless, that perfectly definite numerical value belongs to the point A and therefore to the symbol $\sqrt{2}$. What are t tangible facts which underlie this feeling?

Obviously, in the first instance, this: We do, it is true, know perfectly well that the values 1·4 1·41 or 1·414 etc. for $\sqrt{2}$ are inaccurate, in fact that these (rational) numbers have squares < 2, i. e. a too small. But we also know that the values 1·5 or 1·42 or 1·415 etc. are in the same sense too larg that the value which we are attempting to reach would have therefore to lie between the correspondi too large and too small values. We thus reach the definite conviction that the value of $\sqrt{2}$ is within o grasp, although the given values are all incorrect. The root of this conviction can only lie in the fa that we have at our command a *process,* by which the above values may be continued *as far as w please;* we can, that is, form pairs of decimal fractions, with 1, 2, 3, . . . places of decimals, o fraction of each pair being too large, and the other too small, and the two differing only by one unit the last decimal place, i. e. by $\frac{1}{10^n}$, if n is the number of decimal places. As this difference may b made ***as small as we please,*** by sufficiently increasing the number n of given decimal places, we a taught through the above process to enclose the value which we are in search of between two numbe as near as we please to one another. By a metaphor, somewhat bold at the present stage, we say th through this process V2 itself is "given", — in virtue of it, $\sqrt{2}$ is "known", — by it, $\sqrt{2}$ may b "calculated", and so on.

We have precisely the same situation with regard to any other value which cannot actually b denoted by a rational number, as for instance π, log 2, sin 10° etc. If we say, these numbers are *know* nothing more is implied than that we know some process (in most cases an *extremely* laborious on by which, as detailed in the case of $\sqrt{2}$, the desired value may be imprisoned, hemmed in, within narrower and narrower space between rational numbers, — and this space ultimately narrowed dow as much as we please.

For the purpose of a somewhat more general and more accurate statement of these matters, w insert a discussion of sequences of rational numbers, provisional in character, but nevertheless fundamental importance for all that comes after.

§ 2. Sequences of rational numbers[1].

In the process indicated above for calculating $\sqrt{2}$, successive well-defined rational numbers we constructed; their expression in decimal form was material in the description; from this form we no propose to free it, and start with the following

~~Definition. *If, by means of any suitable process of construction, we can form successively*~~ *a firs a second, a third, . . . (rational) number and if to every positive integer n one and only one wel defined (rational) number* x_n *thus corresponds, then the numbers*

$$x_1,\quad x_2,\quad x_3,\dots, x_n,\dots$$

(in this order, corresponding to the natural order of the integers 1, 2, 3, . . . *n*, ...) *are said to form* ***sequence. We*** *denote it for brevity by* (x_n) *or* $(x_1, x_2 \dots)$.

Examples.

1. $x_n = \frac{1}{n}$; i. e. the sequence $\left(\frac{1}{n}\right)$, or $1, \frac{1}{2}, \frac{1}{3}, \dots, \frac{1}{n}, \dots$,
2. $x_n = 2^n$; i. e. the sequence 2, 4, 8, 16, …
3. $x_n = a^n$; i. e. the sequence $a, a^2, a^3, \dots$, where a is a given number.
4. $x_n = x_n = \frac{1}{2}\{1-(-1)^n\}$; i. e. the sequence 1, 0, 1, 0, 1, 0, . . .
5. x_n = the decimal fraction for $\sqrt{2}$, terminated at the n^{th} digit.
6. $x_n = \frac{(-1)^{n-1}}{n}$ i. e. the sequence 1, — $1, -\frac{1}{2}, +\frac{1}{3}, -\frac{1}{4}, \dots$
7. Let $x_1 = 1, x_2 = 1, x_3 = x_1 + x_2 = 2$ and, generally, for $n \geqq 3$, let $x_n = x_{n-1} + x_{n-2}$. obtain th sequence 1, 1, 2, 3, 5, 8, 13, 21, … , usually called *Fibonacci's* sequence.
8. $1, 2, \frac{1}{2}, -2, -\frac{1}{2}, 3, \frac{1}{3}, -3, -\frac{1}{3}, \dots$
9. $2, \frac{3}{2}, \frac{4}{3}, \frac{5}{4}, \dots, \frac{n+1}{n}, \dots$
10. $0, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \dots, \frac{n-1}{n}, \dots$
11. x_n = the n^{th} prime number[2]; i. e. the sequence 2, 3, 5, 7, 11, 13, …
12. The sequence 1, $1, \frac{3}{2}, \frac{11}{6}, \frac{25}{12}, \frac{137}{60}, \dots$, in which $x_n = x_n = \left(1+\frac{1}{2}+\dots+\frac{1}{n}\right)$.

Remarks.

1. The law of formation may be quite arbitrary; it need not, in particular, be embodied in ar explicit formula enabling us to obtain x_n, for a given *n,* by direct calculation. In examples 6, 5, 7 ar 11, clearly no such formula can be immediately written down. If the terms of the sequence ar individually given, neither the law of formation (cf. 6, 5 and 12) nor any other kind of regularity (cf. 11) among the successive numbers is necessarily apparent.

2. It is sometimes advantageous to start the sequence with a "0^{th}" term x_0, or even with a (— 1) or (— 2)th term, x_{-1}, x_{-2}. Occasionally, it pays better to start indexing with 2 or 3. The only essenti is that there should be an integer $1, \frac{3}{2}, \frac{11}{6}, \frac{25}{12}, \frac{137}{60}, \dots$ such that x_n is defined for every $n \geqq m$. The ter x_m is then called the *initial term* of the sequence. We will however, even then, continue to designate a the n^{th} term that which bears the index *n*. In § 6, 2, 3 and 4, for instance, we can without furth difficulties take a 0^{th} term or even $(—1)^{th}$ or $(—2)^{th}$ to head the sequence. The "first term" of sequence is then not necessarily the term with which the sequence begins. The notation will b preferably $(x_0, x_1 \dots)$ or $(x_{-1}, x_0, \dots)$, etc., as the case may be, unless it is either quite clear