

TOPOS THEORY

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Preface

The origins of this book can be traced back to a series of six seminars, which I gave in Cambridge in the winter of 1973/74, and which formed the nucleus of the present [chapters 1–6](#). Further seminars in the same series, covering parts of [chapters 0, 7 and 9](#), were given by Barry Tension and Robert Seely. By popular request, the notes of these seminars were written up and enjoyed a limited circulation. In the summer of 1974, I began to revise and expand these notes, with the idea that they might some day form a book. During the winter and spring of 1975, whilst at the University of Liverpool, I was able to give a course of lectures covering the material of [chapters 0–5 and 8](#) in some detail. By the end of this period, I had a fairly clear picture of the overall shape of the book; and (encouraged by Michael Butler) I began the actual writing of it in July 1975. From October 1975 to March 1976 I was at the University of Chicago, where there was a weekly seminar on topos theory organized by Saunders Mac Lane and myself; the material covered during this period was drawn mainly from [chapters 2, 4, 5, 6 and 9](#), and the speakers (in addition to myself) were Kathy Edwards, Steve Harris and Steve Landsburg. Also during this period, I wrote the text of [chapters 2–5](#) and most of [chapter 6](#); the remainder of the text was completed during May–July 1976 after my return to Cambridge.

The lectures and seminars mentioned above had a very direct influence on the text of the book, and all those who attended them (in particular those whose names appear above) deserve my thanks for their part they have played in shaping it. But I have also benefited from more informal contacts with many mathematicians at conferences and elsewhere. Among those whose (largely unpublished) ideas I have gladly borrowed are Julian Cole, Radu Diaconescu, Mike Fourman, Peter Freyd, André Joyal and Chris Mulvey. John Gray gave me valuable advice on 2-categorical matters, and Jack Duskin and Barry Tension helped to improve my understanding of cohomology. And I must thank Jean Bénabou for the many ideas I have consciously or inadvertently borrowed from him, and Tim Brook for his help in the compilation of the bibliography.

There remain four mathematicians to whom I owe a debt which must be acknowledged individually. Myles Tierney introduced me to topos theory through his lectures at Varenna in 1971; looking back on the published version [[TV](#)], I still find it incredible that he managed to teach me so much in eight short lectures. Gavin Wraith's help and encouragement have meant a great deal to me, and his Bangor lectures [[WB](#)] served as a model for some parts of this book. Like every other worker in topos theory I owe Bill Lawvere an overwhelming debt in general terms, for his pioneering insights; but I have also benefited at a more personal level from his ideas and conversation. Above all, I have to express my indebtedness to Saunders Mac Lane: but for him I should never have become a topos-theorist in the first place; and the care with which he has read through the original typescript, and provided suggestions for improvement in almost every paragraph, has been altogether out of the ordinary. If there are any major errors or obscurities still remaining in the text, they are surely a testimony to my perversity rather than his lack of vigilance.

On a different, but no less significant, level, I must also thank the Universities of Liverpool and Chicago, and St John's College, Cambridge, for employing me during the writing of the book; Paul Cohn, for accepting it for publication in the L.M.S. Monographs series; and the staff of Academic Press for the efficiency with which they have transformed my amateurish typescript into the book which you see before you.

Cambridge, June 1977

P.T.

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Introduction

Topos theory has its origins in two separate lines of mathematical development, which remained distinct for nearly ten years. In order to have a balanced appreciation of the significance of the subject, I believe it is necessary to consider the history of these two lines, and to understand why they came together when they did. I therefore begin this Introduction with a (personal, and doubtless strongly biased) historical survey.

The earlier of the two lines begins with the rise of *sheaf theory*, originated in 1945 by J. Leray, developed by H. Cartan and A. Weil among others, and culminating in the published work of J. P. Serre [107], A. Grothendieck [42] and R. Godement [TF]. Like a great deal of homological algebra, the theory of sheaves was originally conceived as a tool of algebraic topology, for axiomatizing the notion of “local coefficient system” which was essential for a good cohomology theory of non-simply connected spaces; and the full title of Godement’s book indicates that it was still viewed in this light in 1958. But well before this date, the power of sheaf theory had been recognized by algebraic and analytic geometers; and in more recent years, its influence has spread into many other areas of mathematics. (For two widely-differing examples, see [49] and [106].)

However, in algebraic geometry it was soon discovered that the topological notion of sheaf was not entirely adequate, in that the only topology available on an abstract algebraic variety or scheme, the Zariski topology, did not have “enough open sets” to provide a good geometric notion of localization. In his work on descent techniques [43] and the étale fundamental group [44], A. Grothendieck observed that to replace “Zariski-open inclusion” by “étale morphism” was a step in the right direction; but unfortunately the schemes which are étale over a given scheme do not in general form a partially ordered set. It was thus necessary to invent the notion of “Grothendieck topology” on an arbitrary category, and the generalized notion of sheaf for such a topology, in order to provide a framework for the development of étale cohomology.

This framework was built up during the “Seminaire de Géométrie Algébrique du Bois Marie” held during 1963–64 by Grothendieck with the assistance of M. Artin, J. Giraud, J. L. Verdier and others. (The proceedings of this seminar were published in a revised and greatly enlarged version [GV], including some notable additional results of P. Deligne, eight years later.) Among the most important results of the original seminar was the theorem of Giraud, which showed that the categories of generalized sheaves which arise in this way can be completely characterized by exactness properties and size conditions; in the light of this result, it quickly became apparent that these categories of sheaves were a more important subject of study than the sites (= categories + topologies) which gave rise to them. In view of this, and because a category with a topology was seen as a “generalized topological space”, the (slightly unfortunate) name of *topos* was given to any category satisfying Giraud’s axioms.

Nevertheless, toposes were still regarded primarily as vehicles for carrying cohomology theories, not only étale cohomology, but also the “fppf” and crystalline cohomologies, and others. The power of the machinery developed by Grothendieck was amply demonstrated by the substantial geometric results obtained by using these cohomology theories in the succeeding years, culminating in Deligne’s proof [159] of the famous “Weil conjectures”—the mod- p analogue of the Riemann hypothesis. And the machinery itself was further developed, for example in J. Giraud’s work [38] on nonabelian cohomology. But the full import of the dictum that “the topos is more important than the site” seems never to have been appreciated by the Grothendieck school. For example, though they were aware of the cartesian closed structure of toposes ([GV], IV 10), they never exploited this idea to the full along the lines laid down by Eilenberg–Kelly [160]. It was, therefore, necessary that a second

line of development should provide the impetus for the elementary theory of toposes.

The starting-point of this second line is generally taken to be F. W. Lawvere's pioneering 1969 paper on the elementary theory of the category of sets [71]. However, I believe that it is necessary to go back a little further, to the proof of the Lubkin-Heron-Freyd-Mitchell embedding theorem for abelian categories [AC]. It was this theorem which, by showing that there is an explicit set of elementary axioms which imply all the (finitary) exactness properties of module categories, paved the way for a truly autonomous development of category theory as a foundation for mathematics.

(Incidentally, the Freyd-Mitchell embedding theorem is frequently regarded as a culmination rather than a starting-point; this is because of what seems to me a misinterpretation (or at least an inversion) of its true significance. It is commonly thought of as saying "If you want to prove something about an abelian category, you might as well assume it is a category of modules"; whereas I believe its true import is "If you want to prove something about categories of modules, you might as well work in a general abelian category"—for the embedding theorem ensures that your result will be true in the generality, and by forgetting the explicit structure of module categories you will be forced to concentrate on the essential aspects of the problem. As an example, compare the module-theoretic proof of the Snake Lemma in [HA] with the abelian-category proof in [CW].)

This theorem was soon followed by Lawvere's paper [71], setting out a list of elementary axioms which, with the addition of the non-elementary axioms of completeness and local smallness, are sufficient to characterize the category of sets. (In a subsequent paper [72], Lawvere provided a similar axiomatization of the category of small categories, and D. Schlomiuk [105] did the same for the category of topological spaces.)

One may well ask why this paper was not immediately followed by the explosion of activity which greeted the introduction of elementary toposes six years later. In retrospect, the answer is that Lawvere's axioms were too specialized: the category of sets is an extremely useful object to have as a foundation for mathematics, but as a subject of axiomatic study it is not (*pace* the activities of Martin Solovay *et al.*!) tremendously interesting—it is too "rigid" to have any internal structure. In a similar way, if the abelian-category axioms had applied only to the category of abelian groups, and not to categories of modules or of abelian sheaves, they too would have been neglected. So what was needed for the category of sets was an axiomatization which would also cover set-valued functor categories and categories of set-valued sheaves—i.e. the axioms of an elementary topos.

In his subsequent papers ([73], [75]), Lawvere began to investigate the idea that the two-element set $\{true, false\}$ can be regarded as an "object of truth-values" in the category of sets; in particular, he observed that the presence of such an object in an arbitrary category enables us to reduce the Comprehension Axiom to an elementary statement about adjoint functors. The same idea was at the heart of the work of H. Volger ([125], [126]) on logical and semantical categories.

Meanwhile, the embedding-theorem side of things was advanced by M. Barr [2], who formulated the notion of *exact category* and used it as the basis of a non-additive embedding theorem. The closely-related notion of *regular category* was formulated independently by P. A. Grillet [41] and H. Van Osdol [122], who used it in their investigations of general sheaf theory; and Barr himself observed that Giraud's theorem could be regarded as little more than a special case of his embedding theorem. This perhaps represents (logically, if not chronologically) the first coming-together of the two lines of development mentioned earlier.

However, at about the same time Lawvere's attention also turned towards Grothendieck toposes; he observed that every Grothendieck topos has a truth-value object Ω , and that the notion of Grothendieck topology is closely connected with endomorphisms of Ω (see [LH]). During the years 1969–70, Lawvere and M. Tierney (who had earlier contributed to the theory of exact categories) began to investigate the consequences of taking "there exists an object of truth-values" as an axiom.

the result was elementary topos theory. A remarkably large proportion of the basic theory was developed in that 12-month period, as will be apparent from the large number of theorems in [chapters 1–4](#) of this book whose proof is credited to Lawvere and Tierney.

Once these theorems became known to mathematicians at large (i.e. after Lawvere’s lectures in Zürich and Nice [\[LN\]](#) in the summer of 1970, and the Dalhousie conference [\[LH\]](#) in January 1971) they were immediately taken up and further developed by several people. One of the first and most important was P. Freyd, whose lectures at the University of New South Wales [\[FK\]](#) explored the embedding theory of toposes; in retrospect this seems to have been something of a blind alley, in that the inversion of the usual metatheorem, mentioned above in connection with abelian categories, applies with even more force to topos theory—since the great virtue of the topos axioms is their elementary character, one should not have to appeal to a non-elementary embedding theorem to prove elementary facts about toposes. (Freyd’s embedding theorem will not be found in this book; but the most important (and elementary) part of it, which shows that any topos can be embedded in a Boolean topos, is proved in [§7.5](#).) Nevertheless, Freyd’s work contained a great many important technical results; in particular his characterization of natural number objects is a theorem of major importance.

Amongst other early workers on topos theory, one should mention J. Bénabou and his student Coyleyrette in Paris [\[BC\]](#), and A. Kock and G. C. Wraith in Aarhus [\[KW\]](#). C. J. Mikkelsen, a student of Kock, was the first to prove that one of the Lawvere–Tierney axioms, that of finite colimits, could be deduced from the others; his thesis [\[84\]](#) also contains many important contributions to lattice-theory in a topos.

In view of the Lawvere–Tierney proof of the independence of the continuum hypothesis [\[117\]](#), it became a matter of importance to determine the precise relationship between elementary topos theory and axiomatic set theory. The answer was found independently by J. C. Cole [\[18\]](#), W. Mitchell [\[8\]](#) and G. Osius [\[92\]](#). W. Mitchell also introduced an idea which has since become central to the subject, namely that each topos gives rise to an internal language which can be used to make “quasi-syntactical” statements about objects and morphisms of the topos. Whilst the original idea is due to Mitchell, its most enthusiastic proponent has undoubtedly been J. Bénabou, and his students have used the internal language extensively in recent years.

The next major advance was made by R. Diaconescu, a student of Tierney whose thesis was completed in 1973. Diaconescu’s theorem [\[30\]](#) was important not only for the insight it gave into the 2-categorical structure of \mathbf{Top} , but also because it represented the first significant exploitation of the theory of internal categories. (This theory had developed over the years in a rather haphazard way, largely through unpublished work of J. Bénabou.) As an encore, Diaconescu proved the relative Giraud theorem; Giraud himself [\[39\]](#) had proved a relative version of his theorem (by non-elementary means) for Grothendieck toposes, and W. Mitchell had formulated the correct elementary form. But Mitchell was able to prove this only in the special case when the “object of generators” (see [4.43](#)) is 1; it turned out that Diaconescu’s theorem was the essential tool needed to prove the general case. At about the same time, P. T. Johnstone [\[52\]](#) also used internal categories in his proof that Grothendieck’s construction of the associated sheaf functor could be carried over to the elementary setting.

The next development (which in fact overlapped the previous ones) was the rise of the notion of toposes as theories and the concept of classifying topos. In a sense, this goes right back to Lawvere’s work [\[176\]](#) on algebraic theories, but its connection with topos theory began with the work of M. Hakim [\[45\]](#), a student of Grothendieck, on relative schemes, in the course of which she constructed the classifiers for rings and local rings, and established their fundamental properties. In 1972, A. Joynt and G. E. Reyes [\[RM\]](#) isolated the notion of “coherent theory” (=finitary geometric theory, in our terminology), and proved that every such theory has a classifying topos; their work was later extended by Reyes and M. Makkai [\[82\]](#) to cover infinitary geometric theories.

It was F. W. Lawvere [LB] who first observed that, in view of the work of Joyal and Reyes, the theorem of P. Deligne on points of coherent toposes was precisely equivalent to the Godel-Henkin completeness theorem for finitary geometric theories; and Lawvere too conjectured the “Boolean-valued completeness theorem” for infinitary theories whose topos-theoretic equivalent was proved by M. Barr [4].

Once again, Diaconescu’s theorem provided the key to the “relativization” of the Joyal-Reyes results; the decisive step was taken in 1973 by G. C. Wraith, who constructed an object classifier over an arbitrary topos with a natural number object. From there to the general existence theorem for classifying toposes was little more than a formality; it was achieved independently by A. Joyal, M. Tierney [119] and J. Bénabou [8].

This brings our historical survey up to date, at least where major results are concerned. Now let us consider the present position of topos theory, and its future prospects.

The first thing which must be said is that the basic theory of elementary toposes (i.e. the contents of chapters 1–5 of this book) seems to be almost completely worked out. Indeed, I am aware of only one substantial unanswered question arising from these five chapters (namely the existence of finite (pseudo-)colimits in \mathbf{Top} , touched on in §4.2); doubtless there are many other minor points to be cleared up, and several theorems whose proofs will be improved and simplified in time, but the foundations of the subject do appear to be pretty stable. This is of course a bad thing: it is vital to the health of a subject as basic as topos theory that its fundamental tenets should be the subject of continual review and improvement, and I am uncomfortably aware that by writing this book I have contributed largely to the concreting-over of these foundations. My only defence against this charge is that it seemed to me that the solidification was taking place anyway, and it was better that it should happen in print than in an unpublished folklore accessible only to insiders.

The average mathematician, who regards category theory as “generalized abstract nonsense”, tends to regard topos theory as generalized abstract category theory. (No doubt it has inherited this reputation from its parent, the Grothendieck approach to algebraic geometry.) And yet S. Mac Lane [179] regards the rise of topos theory as a symptom of the *decline* of abstraction in category theory and of abstract algebra in general. I am convinced that Mac Lane is right, and that his insight points the way to the most probable future development of topos theory; for almost all the *recent* work of significance in topos theory has been concerned not with toposes as an abstract and isolated area of mathematics, but with toposes as an aid to understanding and clarifying concepts in other areas. (See for example, [36], [57], [63], [79], [88], [90], [112], [130].)

To take a specific example, consider the general existence theorem for classifying toposes (6.56). One’s first reaction on seeing this theorem is to admire its elegance and generality; the second reaction (which comes quite a long time later) is to realize its fundamental uselessness—a quality which, by the way, it shares with the General Adjoint Functor Theorem. For the only possible use of such a theorem is to reduce the study of a particular geometric theory to the study of its generic model (or conversely, to reduce the study of a particular topos to that of the theory whose generic model it contains), and the theorem as proved in §6.5 simply does not provide an effective means of passing from the one to the other. Thus the “syntactic” proof of the same theorem in §7.4, though appreciably messier, is much more valuable in practice—and it is this proof, not the later one given in the earlier chapter, which has inspired most subsequent work on the subject.

In saying that the future of topos theory lies in the clarification of other areas of mathematics through the application of topos-theoretic ideas, I do not wish to imply that, like Grothendieck, I view topos theory as a machine for the demolition of unsolved problems in algebraic geometry or anywhere else. On the contrary, I think it is unlikely that elementary topos theory itself will solve any major outstanding problems of mathematics; but I do believe that the spreading of the topos-theoretic

outlook into many areas of mathematical activity will inevitably lead to the deeper understanding of the real features of a problem which is an essential prelude to its correct solution.

What, then, is the topos-theoretic outlook? Briefly, it consists in the rejection of the idea that there is a fixed universe of “constant” sets within which mathematics can and should be developed, and the recognition that the notion of “variable structure” may be more conveniently handled within a universe of *continuously variable* sets than by the method, traditional since the rise of abstract set theory, of considering separately a domain of variation (i.e. a topological space) and a succession of constant structures attached to the points of this domain. In the words of F. W. Lawvere [LB], “Even the notion of constancy is relative, being derived perceptually or conceptually as a limiting case of variation, and the undisputed value of such notions in clarifying variation is always limited by their origin. This applies in particular to the notion of constant set, and explains why so much of naïve set theory carries over in some form into the theory of variable sets”. It is this generalization of ideas from constant to variable sets which lies at the heart of topos theory; and the reader who keeps it in mind, as an ultimate objective, whilst reading this book, will gain a great deal of understanding thereby.

Next, a few words on some of the things which I have not done in this book.

(1) In the definition of a topos, I have taken cartesian closedness and the existence of Ω as two separate axioms, instead of combining them into a single axiom of power-objects as suggested by Kock [66]. (The equivalence of Kock’s axiom is, however, covered in the Exercises to [chapter 1](#).) At a practical level, I would defend this decision on two grounds: (a) that there are a number of results in the book (notably in [chapter 2](#)) which use only cartesian closedness and not the full topos axioms, and some (e.g. [Theorem 1.47](#)) where exponentials and Ω are used in essentially different ways in the same proof; and (b) that if one takes the power-object definition, one is obliged (as in [WB]) to follow immediately with the rather technical proof that this definition implies cartesian closedness, and one is in danger of losing one’s readers at this critical point. On a more philosophical level, I would add (c) that the definition via power-objects is really a set-theorist’s rather than a category-theorist’s definition of a topos, in that it subordinates the notion of “function” to that of “subset” by means of the set-theoretic device of identifying functions with their graphs. One of the principal features of category theory is that it takes “morphism” as a primitive notion, on a level with (*not*, incidentally superior to) that of “object”; it is therefore right that the definition of a topos should include its closed structure.

(2) I have not introduced the Mitchell-Bénabou language until rather late in the book, at the end of [chapter 5.1](#). I know that there are some people whose ideal textbook on topos theory would begin with the definition and just enough development of exactness properties to introduce the language and prove the soundness of its interpretation; thereafter all proofs would be conducted within the formal language. I do not agree with this approach; I believe that it is impossible to appreciate the full power of the Mitchell-Bénabou language until you have had some experience of proving things without it (indeed, this is almost the only place in the book where I have consciously followed a particular ordering of material for pedagogical rather than logical reasons). There is also the point that the formal-language approach breaks down when confronted with the relative Giraud [theorem \(4.46\)](#) whilst the Mitchell-Bénabou language is a very powerful tool in proofs within a single topos, it is not well adapted to proofs in which we have to pass back and forth between two toposes by a geometric morphism. (It is possible that the proof of [4.46](#) could be shortened by using the language of local internal categories, but that is a different matter.)

(3) I have already mentioned that Freyd’s embedding theorem [FK] will not be found in this book. In consequence, Freyd’s concept of well-pointed topos plays a relatively minor role; it is not introduced until [§9.3](#).

(4) I have not included any reference to Freyd’s more recent development (unpublished as yet) of the theory of *allegories*. This theory sets out to do for the category of sets and relations what topos theory does for sets and functions; Freyd has been known to maintain that it provides a simpler and more natural basis than topos theory for many of the ideas developed in this book, but I personally remain unconvinced of this.

(5) I have not mentioned the work being done by D. Bourn [13], R. Street [113], [114] and others on the development of a 2-categorical analogue of topos theory. It appears to me, however, that the fundamentals of this theory have not yet reached a sufficiently definitive state for treatment in book form.

(6) One generalization of topos theory whose omission I do slightly regret is J. Penon’s notion of *quasitopos* [99]. However, I feel that to introduce it early in the book would simply have introduced extra complications in the proofs without any benefits in the form of additional well-known examples and to introduce it later on would have involved a good deal of duplication. I hope, nevertheless, that O. Wyler’s forthcoming notes on quasitoposes (promised in [130]) will help to fill this gap.

(7) The phrase “Grothendieck universe” does not appear anywhere in the book. This is intentional; I have deliberately been as vague as possible (except in §9.3) about the features of the set theory which I am using, since it really doesn’t matter. Topos theory is an elementary theory, and its main theorems are not—or ought not to be—dependent on recondite axioms of set theory. (In fact I am a fully paid-up member of the Mathematicians’ Liberation Movement founded by J. H. Conway [157].) If pressed, however, I would admit to using a Gödel-Bernays-type set theory having a distinction between small categories (sets) and large categories (proper classes); but I also wish to consider certain “very large” 2-categories (notably \mathbf{Cat} and \mathbf{Top}) whose objects are themselves large categories. If I wished to be strictly formal about this, I should need to introduce at least one Grothendieck universe; but since all the statements I wish to make about \mathbf{Cat} and \mathbf{Top} are (equivalent to) elementary ones, there is no *real* need to do so. In order to retain some set-theoretic respectability, I have limited myself to considering sheaves only on small sites; this has the disadvantage that we cannot state Giraud’s theorem in its slickest form (a category is a Grothendieck topos iff it is equivalent to the category of canonical sheaves on itself), but is otherwise not as irksome as the authors of [GV] would have us believe.

Finally, I have to state my position on the most controversial question in the whole of topos theory: how to spell the plural of topos. The reader will already have observed that I use the English plural; I do so because (in its mathematical sense) the word topos is not a direct derivative of its Greek root, but a back-formation from topology. I have nothing further to say on the matter, except to ask those toposophers† who persist in talking about topoi whether, when they go out for a ramble on a cold day, they carry supplies of hot tea with them in thermoi.

† I am indebted to Miles Reid for suggesting the terms “toposopher” and “toposophy”: and I urge my fellow-toposophers to adopt them.

Notes for the Reader

Throughout the book, a single numbering system is used for definitions, lemmas, theorems, remarks, etc.; the number $n.pq$ normally denotes the q th numbered reference in the p th paragraph of chapter n , except that in certain paragraphs which have more than nine numbered references, the numbers run on consecutively from $n.p9$. (Thus 8.20 is the tenth numbered reference in §8.1, and 5.5 is the eleventh in §5.4.) Fortunately, it has been possible to do this without overlapping. While the system may not have logic on its side, it does combine the advantage of simple and easily remembered numbers (no profusion of decimal points!) with that of references which are easy to locate. A. n denotes the n th numbered reference in the Appendix.

At the end of each chapter will be found a number of exercises: about ten on each of the earlier chapters, more on the later ones. They vary considerably in difficulty, some being completely routine whereas others are quite substantial. I have not given any indication of which exercises I consider to be easier (the order is that of the material in the preceding chapter to which they refer), but I have given fairly copious hints for most of the harder ones. In quite a number of cases, the result of an exercise is used in either the exercises or the text of a subsequent chapter; these exercises are distinguished by an obelus (†).

The following summary of the logical interdependence of the various chapters may be useful to the reader who is interested in one particular topic. Chapter 0 contains a summary of certain background material which is required either to motivate the definition of a topos, or to provide a source of examples. Chapters 1–5 form the core of the book; of these chapters 1–4 follow a more or less geodesic path (with a few digressions such as §4.2) from the definition of a topos (1.11) to the relative Giraud theorem (4.46) and the existence of pullbacks in $\mathcal{B}\text{Top}/\mathcal{E}$ (4.48). The logical dependence relation in these four chapters is thus fairly close to being a total order.

However, the majority of the material in chapter 2 (on internal categories) is fairly technical, and some readers may find it fairly difficult at a first reading. I would advise such readers to omit the whole of chapter 2, except for Theorem 2.32 (which is important, and has applications in other areas than internal category theory), and go on to chapter 3. (There are some references to chapter 2 in §3.1, but you can refer back for these as necessary.) You may then go on to the first paragraph (only) of chapter 4, the whole of chapter 5 except for some parts of §5.3, and even the first two paragraphs of chapter 6, before returning to tackle chapter 2.

Chapter 5 introduces a number of concepts which, although part of the mainstream of topos theory, are not involved in the proof of the relative Giraud theorem. In particular, it contains a description of the internal language of a topos, which is used freely in the second half of the book.

The last four chapters present various extensions and applications of the basic theory; I originally hoped to make them logically independent, so that they could be read in any order, but inevitably some cross-connections have established themselves. The following table summarizes the important ones:

<i>Before reading</i>	<i>it is advisable to read</i>
7.4	6.3 and 6.5
8.1	7.5
8.4	6.2
9.1	6.2 and 6.4

There are some further cross-connections between the exercises on these four chapters (see, for example, [exercises 6.11, 8.7, 9.6 and 9.14](#)).

The Appendix is a presentation of material originally intended for inclusion in [chapter 2](#); it was removed from there because, whilst it seems certain to become part of the mainstream of topos theory before very long, it is possible that the basic definition of “locally internal category” has not yet reached its final form. It may be read at any time after [chapter 2](#), although it does make a number of references to later chapters.

Throughout the book, references to the bibliography are enclosed in square brackets. The bibliography itself is divided into four sections: section A consists of “standard references” to other areas of mathematics (e.g. lattice theory, algebraic topology) which are used when a theorem or definition from one of these areas is quoted in the text. Section B contains works of a general nature on topos theory, and some introductory articles written for non-specialist readers (e.g. [\[MB\]](#), [\[WI\]](#)). Section C contains the remaining references on topos theory, and a number of closely-related papers on category theory, sheaf theory, etc. Sections B and C together aim to present a complete list of articles so far published on topos theory; however, I do not include short abstracts of talks, nor Ph.D. theses unless they contain important results not published elsewhere. Section D contains the remaining papers which are referred to in the text. References in sections A and B are indicated by two-letter codes; those in sections C and D are consecutively numbered. In all four sections, I have indicated the *Mathematical Reviews* review numbers, where they exist.

TOPOS THEORY

Preliminaries

0.1. CATEGORY THEORY

This paragraph is not intended to give a detailed introduction to the basic ideas of category theory; our intention is rather to indicate some of the concepts and theorems which we shall be assuming familiar, and to set out some standard notations. The reader who considers himself unfamiliar with these basic concepts would be well advised to refer to the excellent book by Mac Lane [CW], or to any other standard text on category theory, before proceeding further with this book.

We shall normally use script capitals ($\mathcal{C}, \mathcal{D}, \mathcal{E}, \dots$) to denote “large” categories. When we make the assertion “ \mathcal{C} is a category”, without further qualification, we mean that \mathcal{C} is a model of the “elementary theory of categories” [72], i.e. \mathcal{C} is a *metacategory* in the sense of [CW], chapter I. This means that we do not regard \mathcal{C} as being formally defined within a particular model of set theory; in particular, if X and Y are objects of \mathcal{C} , we do not require that the morphisms of \mathcal{C} from X to Y shall form a set.

However, except in [chapter 9](#), we shall normally assume that we are given a (fixed) model of some suitable set theory (including the axiom of choice when necessary); and we shall use the letter \mathcal{S} to denote the *category of sets and functions* which we obtain from it. We use the term *small category* for a category whose morphisms form a set. If C is a small category, we write $\mathcal{S}^{C^{\text{op}}}$ for the category of *presheaves* on C , i.e. contravariant functors from C to \mathcal{S} ; amongst the objects of $\mathcal{S}^{C^{\text{op}}}$, we have the *representable* functors h_U , where U is an object of C , defined by $h_U(V) = \text{hom}_C(V, U)$. (For typographical reasons, we shall sometimes write $h(U)$ for h_U ; we shall also write h^U for the covariant representable functor $\text{hom}_C(U, -)$.) We shall make frequent use of the following two results:

0.11 LEMMA (Yoneda [187]). *For objects U, X of C and $\mathcal{S}^{C^{\text{op}}}$ respectively, there is a bijection (natural in both variables) between morphisms $h_U \longrightarrow X$ in $\mathcal{S}^{C^{\text{op}}}$ and elements of the set $X(U)$. ■*

0.12 LEMMA. *Any object of $\mathcal{S}^{C^{\text{op}}}$ can be expressed as a colimit of a diagram whose vertices are representable functors.*

Proof Let X be an object of $\mathcal{S}^{C^{\text{op}}}$, and let $(C \downarrow X)$ denote the (small) *comma category* whose objects are pairs (U, α) , U an object of C and $\alpha: h_U \longrightarrow X$ in $\mathcal{S}^{C^{\text{op}}}$, and whose morphisms are commutative triangles

$$\begin{array}{ccc} h_U & \xrightarrow{\quad} & h_V \\ & \searrow & \swarrow \\ & & X \end{array}$$

in \mathcal{C}^{cop} .
 Then we have an obvious “forgetful” diagram $(\mathbf{C} \downarrow X) \longrightarrow \mathcal{C}^{\text{cop}}$ given by $(U, \alpha) \longmapsto h_U$; and the colimit of this diagram is readily seen to be X . ■

We assume that the reader is familiar with the notions of *limit* and *colimit*, and of *adjoint functors*. (We use the notation $T \dashv G$ for “ T is left adjoint to G ”.) A word of warning here: when we say that a category *has limits* of a particular type, we mean that there is given, for each diagram of the appropriate type, a *canonical* choice of limit. Thus for example, when we say that \mathcal{C} has binary products we mean that, given sets X and Y , there exists not merely a set whose elements are in 1-1 correspondence with ordered pairs $\langle x, y \rangle$, but a canonical such set, namely the set of all ordered pairs, itself. However, when we say that a functor *preserves limits*, we do not imply that it preserves the canonical choice of limits; and the statement that a given object is *a* limit of a certain diagram does not imply that it is the canonical one.

If a category \mathcal{C} has a terminal object (i.e. a limit for the empty diagram), we denote it by 1 ; and if X is any object of \mathcal{C} , we use 1_X also to denote the unique morphism $X \longrightarrow 1$, and 1_X (or simply 1) for the identity morphism on X . (The confusion inherent in this notation may be loosely justified by the fact that 1_X is the terminal object of the category \mathcal{C}/X of *objects over X* , and that if $Y \xrightarrow{f} X$ is an object of this category, f is also the unique morphism of \mathcal{C}/X from f to 1_X .) If \mathcal{C} has products and/or pullbacks, we use the letter π to denote the canonical projection of a product or pullback onto one of its factors, with suffix 1, 2, 3, . . . to denote the first, second, third, . . . factor. Similarly, we shall usually (but not exclusively) use the letter ν , with appropriate suffix, to denote the inclusion of one factor in a coproduct.

We also assume familiarity with the notions of *monad* (or *triple*) and *comonad*, and of *algebras over a monad*. We shall make use of the “crude tripleability theorem” of Beck [153] in its “reflexive coequalizer” form; recall that a parallel pair $X \begin{matrix} \xrightarrow{f} \\ \xrightarrow{g} \end{matrix} X$ in a category \mathcal{C} is said to be *reflexive* if there exists $Y \xrightarrow{h} X$ such that $fh = gh = 1_Y$. [In the case $\mathcal{C} = \mathcal{S}$, this is equivalent to saying that the image of $X \xrightarrow{(f,g)} Y \times Y$ is a reflexive binary relation on Y .]

0.13 THEOREM. Let $\mathcal{C} \begin{matrix} \xrightarrow{F} \\ \xleftarrow{U} \end{matrix} \mathcal{A}$ be a pair of functors with $F \dashv U$, and let \mathbb{H} be the monad on \mathcal{C} induced by this adjunction. Suppose that \mathcal{A} has coequalizers of reflexive pairs, that U preserves them, and that U reflects isomorphisms. Then U is monadic; i.e. the comparison functor $K: \mathcal{A} \longrightarrow \mathcal{C}^{\mathbb{H}}$ is an equivalence of categories, where $\mathcal{C}^{\mathbb{H}}$ denotes the category of \mathbb{H} -algebras. ■

We shall also make use of the following theorems about categories of algebras:

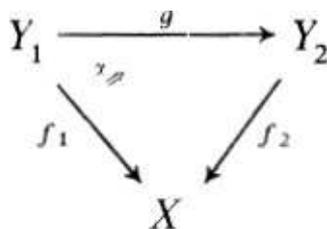
0.14 THEOREM (Eilenberg–Moore [162]). Let $\mathbb{H} = (H, \eta, \mu)$ be a monad on \mathcal{C} , and suppose H has a right adjoint G . Then there is a unique comonad structure $\mathbb{G} = (G, \varepsilon, \delta)$ on G , such that the category $\mathcal{C}_{\mathbb{G}}$ of \mathbb{G} -coalgebras is isomorphic to $\mathcal{C}^{\mathbb{H}}$, by an isomorphism which identifies the two forgetful functors. ■

0.15 THEOREM (“Adjoint lifting theorem”; see [54]). Let \mathbb{H}, \mathbb{K} be monads on categories \mathcal{C}, \mathcal{D} , respectively, $T: \mathcal{C} \longrightarrow \mathcal{D}$ a functor, and $\bar{T}: \mathcal{C}^{\mathbb{H}} \longrightarrow \mathcal{D}^{\mathbb{K}}$ a functor which is a “lifting” of T in the

So the lower right-hand square is a pushout; i.e. a morphism $Y_2 \xrightarrow{\theta} T$ factors through $Y_2 \xrightarrow{\gamma} Z_3$ iff it coequalizes both (g_1, g_2) and (β_1, β_2) . But if this condition is satisfied, then $\theta\beta_1 = \theta\beta_2f_1 = \theta g_1\alpha_2 = \theta g_2\alpha_2 = \theta\beta_2f_2$. Conversely, if $\theta\beta_1f_1 = \theta\beta_2f_2$ and $X_2 \xrightarrow{s} X_1$ is a common splitting for f_1 and f_2 , then $\theta\beta_1 = \theta\beta_1f_1s = \theta\beta_2f_2s = \theta\beta_2$; and similarly $\theta g_1 = \theta g_2$. So $Y_2 \xrightarrow{\gamma} Z_3$ is a coequalizer of β_1f_1 and β_2f_2 . ■

In conclusion, we must mention two areas of category theory not covered in [CW]. One is the theory of 2-categories: a 2-category \mathfrak{C} is a category whose “hom-sets” have the structure of (not necessarily small) categories. That is to say for each pair of objects (X, Y) of \mathfrak{C} , we have a category $\mathfrak{C}(X, Y)$ whose objects are the morphisms (or 1-arrows) of \mathfrak{C} from X to Y , and whose morphisms are called 2-arrows of \mathfrak{C} ; and the operation of composing 1-arrows of \mathfrak{C} is functorial in both variables. Introductions to the theory of 2-categories will be found in [155], [167] and [172].

The reader is warned that, when discussing 2-categories, we shall normally use terms such as “functor” and “limit”, and notations such as \mathfrak{C}/X , in what the Australian school [172] would call the “pseudo” sense, in which diagrams which commute “on the nose” are replaced by diagrams commuting up to a (specified) 2-isomorphism. For example, \mathfrak{C}/X is the 2-category whose objects are 1-arrows $Y \xrightarrow{f} X$ of \mathfrak{C} , and whose 1-arrows are triangles



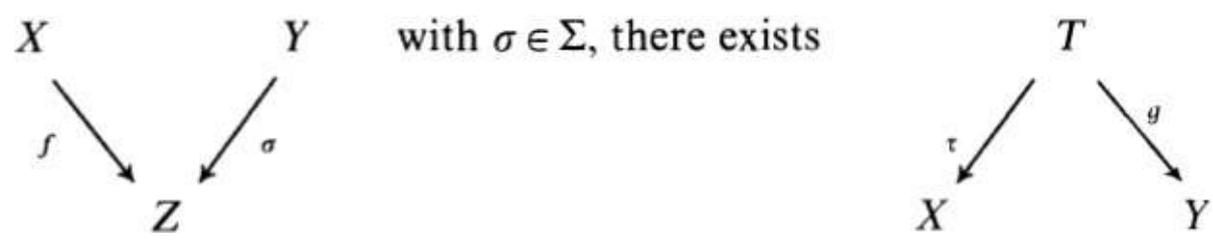
commuting up to a specified 2-isomorphism α ; a 2-arrow of \mathfrak{C}/X from (g_1, α_1) to (g_2, α_2) is a 2-arrow $g_1 \xrightarrow{\beta} g_2$ of \mathfrak{C} such that $(f_2 * \beta)\alpha_1 = \alpha_2$. Similarly, when we speak of a functor $T: \mathfrak{C} \rightarrow \mathfrak{D}$ between 2-categories, we do not imply that T commutes exactly with composition of 1-arrows, but only up to coherent natural 2-isomorphism. (It is understood that if \mathcal{C} is an ordinary category, we identify it with the *locally discrete* 2-category whose objects and 1-arrows are the objects and morphisms of \mathcal{C} , and whose only 2-arrows are identities.)

Our reason for departing from the Australian convention on nomenclature is that it is the “pseudo” concepts which arise most frequently in practice. If we wish to emphasize the fact that a particular functor commutes exactly with composition of 1-arrows (i.e. is a functor in the Australian sense), we shall call it a *strict functor*. In one paragraph (4.2), we shall have occasion to consider the still weaker notion of *lax functor*, in which the 2-arrows up to which diagrams commute need not even be invertible; we shall define these lax concepts explicitly when we need them. And in §2.4 we shall encounter an example of a *bicategory*, which is a “pseudo-2-category” in the Australian sense; i.e. it is defined by the same data as a 2-category, but the unitary and associative laws for composition of arrows hold only up to coherent natural 2-isomorphism.

Finally, we must introduce the notion of category of fractions. If \mathcal{C} is a category and Σ a class of morphisms of \mathcal{C} , then the *category of fractions* $\mathcal{C}\Sigma^{-1}$ is defined (up to equivalence) by requiring that there exist a functor $P_\Sigma: \mathcal{C} \rightarrow \mathcal{C}\Sigma^{-1}$ which is universal among functors $\mathcal{C} \rightarrow \mathcal{D}$ sending all the morphisms in Σ to isomorphisms.

0.18 DEFINITION. A class Σ of morphisms of \mathcal{C} is said to admit a *calculus of right fractions* if:

- (i) Σ is closed under composition, and contains all identity morphisms of \mathcal{C} .
- (ii) Given a diagram

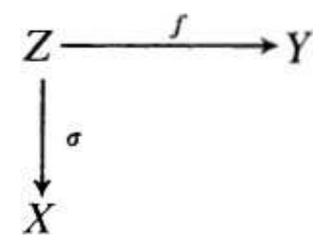


with $\tau \in \Sigma$, such that $f\tau = \sigma g$. (Note: T need not be a pullback.)

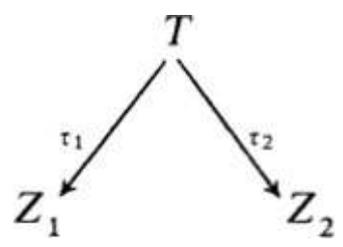
- (iii) Given a diagram $X \begin{matrix} \xrightarrow{f} \\ \xrightarrow{g} \end{matrix} Y \xrightarrow{\sigma} Z$ with $\sigma \in \Sigma$, $\sigma f = \sigma g$, there exists $T \xrightarrow{\tau} X$ in Σ with $\tau f = g\tau$. ■

0.19 THEOREM (Gabriel–Zisman [CF]). Let \mathcal{C} be a category with finite limits, and Σ a class of morphisms of \mathcal{C} admitting a calculus of right fractions. Then the category of fractions $\mathcal{C}\Sigma^{-1}$ has finite limits, and P_Σ preserves them.

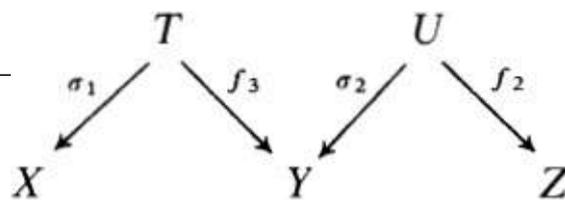
Proof. We show that $\mathcal{C}\Sigma^{-1}$ can be described as follows: its objects are those of \mathcal{C} , and morphisms $X \longrightarrow Y$ in $\mathcal{C}\Sigma^{-1}$ are equivalence classes of diagrams



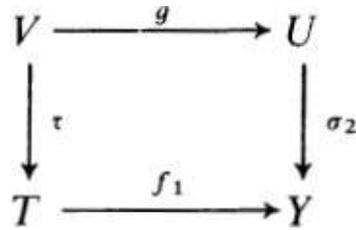
in $\mathcal{C}(\sigma \in \Sigma)$, under the relation that (σ_1, f_1) is equivalent to (σ_2, f_2) iff there exists



such that $\sigma_1\tau_2 = \sigma_2\tau_1 \in \Sigma$ and $f_1\tau_2 = f_2\tau_1$. (It is not hard to verify that this does define an equivalence relation.) To compose two morphisms



in \mathcal{C}^Σ^{-1} , we form a commutative square



with $\tau \in \Sigma$ (using 0.18(ii)), and then define $(\sigma_2, f_2)(\sigma_1, f_1) = (\sigma_1 \tau, f_2 g)$. Again it is easy to check that this definition is independent of the choice of representatives (σ_i, f_i) and of the choice of V ; hence also the composition it defines is associative. The functor P_Σ is defined by

$$P_\Sigma(X) = X, P_\Sigma(X \xrightarrow{f} Y) = \begin{array}{ccc} X & \xrightarrow{f} & Y \\ & \downarrow 1 & \\ & X & \end{array} ;$$

its universal property is immediate. We now show that \mathcal{C}^Σ^{-1} has equalizers and P_Σ preserves them. (The argument for finite products is similar but easier.) Let

$$\begin{array}{ccc}
 Z_i & \xrightarrow{f_i} & Y \quad (i = 1, 2) \\
 \downarrow \sigma_i & & \\
 X & &
 \end{array}$$

be a parallel pair of morphisms of \mathcal{C}^Σ^{-1} ; then by a suitable choice of representatives we may ensure that $Z_1 = Z_2$ and $\sigma_1 = \sigma_2$. Let $E \xrightarrow{q} Z$ be the equalizer of f_1 and f_2 in C . Now let

$$\begin{array}{ccc}
 U & \xrightarrow{g} & X \\
 \downarrow \tau & & \\
 T & &
 \end{array}$$

be a morphism of \mathcal{C}^Σ^{-1} equalizing (σ, f_1) and (σ, f_2) ; then for a suitable choice of commutative square

$$\begin{array}{ccc}
 V & \xrightarrow{h} & Z \\
 \downarrow \rho & & \downarrow \sigma \\
 U & \xrightarrow{g} & X
 \end{array}$$

with $\tau\rho \in \Sigma$ we can ensure that $f_1h = f_2h$ in C , and so there exists a factorization $V \xrightarrow{k} E$ of (ρ, g) through q . Now

$$\begin{array}{ccc}
 V & \xrightarrow{k} & E \\
 \downarrow \tau\rho & & \\
 T & &
 \end{array}$$

is a factorization of (τ, g) through $P_\Sigma(\sigma q)$ in $\mathcal{C}\Sigma^{-1}$; and it is not hard to verify that it is unique. Since $P_\Sigma(\sigma q)$ is an equalizer of (σ, f_1) and (σ, f_2) in $\mathcal{C}\Sigma^{-1}$; hence also P_Σ preserves equalizers. ■

0.2. SHEAF THEORY

As in the previous paragraph, we do not intend to give a detailed account of the “classical” theory of sheaves on a topological space; for this the reader is referred to one of the standard texts on the subject, for example [ST] or [TF]. However, a brief review of the classical theory is indispensable as an aid to understanding the motivation for the more general theory of Grothendieck topologies which follows in the next paragraph.

Let (X, \mathbf{T}) be a topological space. The set \mathbf{T} of open subsets of X is partially ordered by inclusion, and we can regard it as a small category in the usual way; i.e. the objects of \mathbf{T} are the open sets, and \mathbf{T} has one morphism from U to V iff $U \subseteq V$.

0.21 DEFINITION. A *presheaf* (of sets) on X is a presheaf on the category \mathbf{T} , i.e. a contravariant functor from \mathbf{T} to \mathcal{S} . A presheaf P is thus specified by giving a set $P(U)$ for each open $U \subseteq X$, and a *restriction map*

$$\rho_V^U: P(U) \longrightarrow P(V)$$

for each $V \subseteq U$, subject to the obvious compatibility conditions. A *morphism of presheaves* is just a natural transformation of functors. ■

The notion of presheaf includes many examples which are familiar concepts in elementary topology.

0.22 EXAMPLES. (i) For any set A , we have the *constant presheaf* \bar{A} defined by $\bar{A}(U) = A$, $\rho_V^U = 1_A$ for all $V \subseteq U \subseteq X$.

(ii) For any open $U \subseteq X$, we have the representable presheaf h_U , defined by

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