TOPOS THEORY

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Preface

The origins of this book can be traced back to a series of six seminars, which I gave in Cambrid in the winter of 1973/74, and which formed the nucleus of the present chapters 1–6. Further semina in the same series, covering parts of chapters 0, 7 and 9, were given by Barry Tennison and Robe Seely. By popular request, the notes of these seminars were written up and enjoyed a limite circulation. In the summer of 1974, I began to revise and expand these notes, with the idea that th might some day form a book. During the winter and spring of 1975, whilst at the University Liverpool, I was able to give a course of lectures covering the material of chapters 0–5 and 8 in son detail. By the end of this period, I had a fairly clear picture of the overall shape of the book; ar (encouraged by Michael Butler) I began the actual writing of it in July 1975. From October 1975 March 1976 I was at the University of Chicago, where there was a weekly seminar on topos theo organized by Saunders Mac Lane and myself; the material covered during this period was draw mainly from chapters 2, 4, 5, 6 and 9, and the speakers (in addition to myself) were Kathy Edward Steve Harris and Steve Landsburg. Also during this period, I wrote the text of chapters 2–5 and mo of chapter 6; the remainder of the text was completed during May-July 1976 after my return Cambridge.

The lectures and seminars mentioned above had a very direct influence on the text of the book, ar all those who attended them (in particular those whose names appear above) deserve my thanks for th part they have played in shaping it. But I have also benefited from more informal contacts with mar mathematicians at conferences and elsewhere. Among those whose (largely unpublished) ideas I hav gladly borrowed are Julian Cole, Radu Diaconescu, Mike Fourman, Peter Freyd, André Joyal ar Chris Mulvey. John Gray gave me valuable advice on 2-categorical matters, and Jack Duskin ar Barry Tennison helped to improve my understanding of cohomology. And I must thank Jean Bénabo for the many ideas I have consciously or inadvertently borrowed from him, and Tim Brook for h help in the compilation of the bibliography.

There remain four mathematicians to whom I owe a debt which must be acknowledged individuall Myles Tierney introduced me to topos theory through his lectures at Varenna in 1971; looking back o the published version [TV], I still find it incredible that he managed to teach me so much in eig short lectures. Gavin Wraith's help and encouragement have meant a great deal to me, and his Bang lectures [WB] served as a model for some parts of this book. Like every other worker in topos theor I owe Bill Lawvere an overwhelming debt in general terms, for his pioneering insights; but I have als benefited at a more personal level from his ideas and conversation. Above all, I have to express m indebtedness to Saunders Mac Lane: but for him I should never have become a topos-theorist in th first place; and the care with which he has read through the original typescript, and provide suggestions for improvement in almost every paragraph, has been altogether out of the ordinary. there are any major errors or obscurities still remaining in the text, they are surely a testimony to m perversity rather than his lack of vigilance.

On a different, but no less significant, level, I must also thank the Universities of Liverpool ar Chicago, and St John's College, Cambridge, for employing me during the writing of the book; Pa Cohn, for accepting it for publication in the L.M.S. Monographs series; and the staff of Academ Press for the efficiency with which they have transformed my amateurish typescript into the boo which you see before you.

Cambridge, June 1977 P.T.

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Introduction

Topos theory has its origins in two separate lines of mathematical development, which remaine distinct for nearly ten years. In order to have a balanced appreciation of the significance of the subjec I believe it is necessary to consider the history of these two lines, and to understand why they can together when they did. I therefore begin this Introduction with a (personal, and doubtless strong biased) historical survey.

The earlier of the two lines begins with the rise of *sheaf theory*, originated in 1945 by J. Lera developed by H. Cartan and A. Weil among others, and culminating in the published work of J. Serre [107], A. Grothendieck [42] and R. Godement [TF]. Like a great deal of homological algebr the theory of sheaves was originally conceived as a tool of algebraic topology, for axiomatizing th notion of "local coefficient system" which was essential for a good cohomology theory of non-simpl connected spaces; and the full title of Godement's book indicates that it was still viewed in this lig in 1958. But well before this date, the power of sheaf theory had been recognized by algebraic an analytic geometers; and in more recent years, its influence has spread into many other areas mathematics. (For two widely-differing examples, see [49] and [106].)

However, in algebraic geometry it was soon discovered that the topological notion of sheaf was n entirely adequate, in that the only topology available on an abstract algebraic variety or scheme, th Zariski topology, did not have "enough open sets" to provide a good geometric notion of localizatio In his work on descent techniques [43] and the étale fundamental group [44], A. Grothendiec observed that to replace "Zariski-open inclusion" by "étale morphism" was a step in the rig direction; but unfortunately the schemes which are étale over a given scheme do not in general form partially ordered set. It was thus necessary to invent the notion of "Grothendieck topology" on a arbitrary category, and the generalized notion of sheaf for such a topology, in order to provide framework for the development of étale cohomology.

This framework was built up during the "Seminaire de Géometrie Algébrique du Bois Marie" he during 1963–64 by Grothendieck with the assistance of M. Artin, J. Giraud, J. L. Verdier and other (The proceedings of this seminar were published in a revised and greatly enlarged version [GV including some notable additional results of P. Deligne, eight years later.) Among the most importan results of the original seminar was the theorem of Giraud, which showed that the categories generalized sheaves which arise in this way can be completely characterized by exactness properti and size conditions; in the light of this result, it quickly became apparent that these categories sheaves were a more important subject of study than the sites (= categories + topologies) which gav rise to them. In view of this, and because a category with a topology was seen as a "generalize topological space", the (slightly unfortunate) name of *topos* was given to any category satisfyin Giraud's axioms.

Nevertheless, toposes were still regarded primarily as vehicles for carrying cohomology theoreie not only étale cohomology, but also the "fppf" and crystalline cohomologies, and others. The power the machinery developed by Grothendieck was amply demonstrated by the substantial geometric results obtained by using these cohomology theories in the succeeding years, culminating in Deligne's proof [159] of the famous "Weil conjectures"—the mod-p analogue of the Rieman hypothesis. And the machinery itself was further developed, for example in J. Giraud's work [38] nonabelian cohomology. But the full import of the dictum that "the topos is more important than th site" seems never to have been appreciated by the Grothendieck school. For example, though the were aware of the cartesian closed structure of toposes ([GV], IV 10), they never exploited this idea the full along the lines laid down by Eilenberg–Kelly [160]. It was, therefore, necessary that a secon line of development should provide the impetus for the elementary theory of toposes.

~~The starting-point of this second line is generally taken to be F. W. Lawvere's pioneering~~ 196 paper on the elementary theory of the category of sets [71]. However, I believe that it is necessary t go back a little further, to the proof of the Lubkin-Heron-Freyd-Mitchell embedding theorem fo abelian categories [AC]. It was this theorem which, by showing that there is an explicit set elementary axioms which imply all the (finitary) exactness properties of module categories, paved th way for a truly autonomous development of category theory as a foundation for mathematics.

(Incidentally, the Freyd-Mitchell embedding theorem is frequently regarded as a culmination rath than a starting-point; this is because of what seems to me a misinterpretation (or at least an inversio of its true significance. It is commonly thought of as saying "If you want to prove something about a abelian category, you might as well assume it is a category of modules"; whereas I believe its tru import is "If you want to prove something about categories of modules, you might as well work in general abelian category"—for the embedding theorem ensures that your result will be true in th generality, and by forgetting the explicit structure of module categories you will be forced t concentrate on the essential aspects of the problem. As an example, compare the module-theoret proof of the Snake Lemma in [HA] with the abelian-category proof in [CW].)

This theorem was soon followed by Lawvere's paper [71], setting out a list of elementary axion which, with the addition of the non-elementary axioms of completeness and local smallness, a sufficient to characterize the category of sets. (In a subsequent paper [72], Lawvere provided a simil axiomatization of the category of small categories, and D. Schlomiuk [105] did the same for th category of topological spaces.)

One may well ask why this paper was not immediately followed by the explosion of activity whic greeted the introduction of elementary toposes six years later. In retrospect, the answer is th Lawvere's axioms were too specialized: the category of sets is an extremely useful object to have as foundation for mathematics, but as a subject of axiomatic study it is not (*pace* the activities of Marti Solovay *et al.*!) tremendously interesting—it is too "rigid" to have any internal structure. In a simil way, if the abelian-category axioms had applied only to the category of abelian groups, and not t categories of modules or of abelian sheaves, they too would have been neglected. So what was neede for the category of sets was an axiomatization which would also cover set-valued functor categori and categories of set-valued sheaves—i.e. the axioms of an elementary topos.

In his subsequent papers ([73], [75]), Lawvere began to investigate the idea that the two-element s {*true*, *false*} can be regarded as an "object of truth-values" in the category of sets; in particular, h observed that the presence of such an object in an arbitrary category enables us to reduce th Comprehension Axiom to an elementary statement about adjoint functors. The same idea was at th heart of the work of H. Volger ([125], [126]) on logical and semantical categories.

Meanwhile, the embedding-theorem side of things was advanced by M. Barr [2], who formulate the notion of *exact category* and used it as the basis of a non-additive embedding theorem. Th closely-related notion *of regular category* was formulated independently by P. A. Grillet [41] and D H. Van Osdol [122], who used it in their investigations of general sheaf theory; and Barr himse observed that Giraud's theorem could be regarded as little more than a special case of his embeddin theorem. This perhaps represents (logically, if not chronologically) the first coming-together of th two lines of development mentioned earlier.

However, at about the same time Lawvere's attention also turned towards Grothendieck toposes; h observed that every Grothendieck topos has a truth-value object Ω, and that the notion Grothendieck topology is closely connected with endomorphisms of Ω (see [LH]). During the ye 1969–70, Lawvere and M. Tierney (who had earlier contributed to the theory of exact categorie began to investigate the consequences of taking "there exists an object of truth-values" as an axion the result was elementary topos theory. A remarkably large proportion of the basic theory wa ~~developed in that 12-month period, as will be apparent from the large number of theorems in chapte~~ 1–4 of this book whose proof is credited to Lawvere and Tierney.

Once these theorems became known to mathematicians at large (i.e. after Lawvere's lectures Zürich and Nice [LN] in the summer of 1970, and the Dalhousie conference [LH] in January 1971 they were immediately taken up and further developed by several people. One of the first and mo important was P. Freyd, whose lectures at the University of New South Wales [FK] explored th embedding theory of toposes; in retrospect this seems to have been something of a blind alley, in th the inversion of the usual metatheorem, mentioned above in connection with abelian categorie applies with even more force to topos theory—since the great virtue of the topos axioms is the elementary character, one should not have to appeal to a non-elementary embedding theorem to prov elementary facts about toposes. (Freyd's embedding theorem will not be found in this book; but th most important (and elementary) part of it, which shows that any topos can be embedded in a Boolea topos, is proved in §7.5.) Nevertheless, Freyd's work contained a great many important technic results; in particular his characterization of natural number objects is a theorem of major importance

Amongst other early workers on topos theory, one should mention J. Bénabou and his student Celeyrette in Paris [BC], and A. Kock and G. C. Wraith in Aarhus [KW]. C. J. Mikkelsen, a student Kock, was the first to prove that one of the Lawvere–Tierney axioms, that of finite colimits, could b deduced from the others; his thesis [84] also contains many important contributions to lattice-theo in a topos.

In view of the Lawvere–Tierney proof of the independence of the continuum hypothesis [117], became a matter of importance to determine the precise relationship between elementary topos theo and axiomatic set theory. The answer was found independently by J. C. Cole [18], W. Mitchell [8 and G. Osius [92]. W. Mitchell also introduced an idea which has since become central to the subjec namely that each topos gives rise to an internal language which can be used to make "quasi-se theoretical" statements about objects and morphisms of the topos. Whilst the original idea is due Mitchell, its most enthusiastic proponent has undoubtedly been J. Bénabou, and his students have use the internal language extensively in recent years.

The next major advance was made by R. Diaconescu, a student of Tierney whose thesis wa completed in 1973. Diaconescu's theorem [30] was important not only for the insight it gave into th 2-categorical structure of $\mathfrak{Top}$, but also because it represented the first significant exploitation of th theory of internal categories. (This theory had developed over the years in a rather haphazard wa largely through unpublished work of J. Bénabou.) As an encore, Diaconescu proved the relative Girau theorem; Giraud himself [39] had proved a relative version of his theorem (by non-elementary mean for Grothendieck toposes, and W. Mitchell had formulated the correct elementary form. But Mitche was able to prove this only in the special case when the "object of generators" (see 4.43) is 1; it turne out that Diaconescu's theorem was the essential tool needed to prove the general case. At about th same time, P. T. Johnstone [52] also used internal categories in his proof that Grothendieck construction of the associated sheaf functor could be carried over to the elementary setting.

The next development (which in fact overlapped the previous ones) was the rise of the notion toposes as theories and the concept of classifying topos. In a sense, this goes right back to Lawvere work [176] on algebraic theories, but its connection with topos theory began with the work of M Hakim [45], a student of Grothendieck, on relative schemes, in the course of which she constructed th classifiers for rings and local rings, and established their fundamental properties. In 1972, A. Joy and G. E. Reyes [RM] isolated the notion of "coherent theory" (=finitary geometric theory, in ou terminology), and proved that every such theory has a classifying topos; their work was later extende by Reyes and M. Makkai [82] to cover infinitary geometric theories.

It was F. W. Lawvere [LB] who first observed that, in view of the work of Joyal and Reyes, th ~~theorem of P. Deligne on points of coherent toposes was precisely equivalent to the Godel-Henk~~ completeness theorem for finitary geometric theories; and Lawvere too conjectured the “Boolea valued completeness theorem” for infinitary theories whose topos-theoretic equivalent was proved b M. Barr [4].

Once again, Diaconescu’s theorem provided the key to the “relativization” of the Joyal-Reye results; the decisive step was taken in 1973 by G. C. Wraith, who constructed an object classifier ov an arbitrary topos with a natural number object. From there to the general existence theorem f classifying toposes was little more than a formality; it was achieved independently by A. Joyal, M Tierney [119] and J. Bénabou [8].

This brings our historical survey up to date, at least where major results are concerned. Now let u consider the present position of topos theory, and its future prospects.

The first thing which must be said is that the basic theory of elementary toposes (i.e. the contents chapters 1–5 of this book) seems to be almost completely worked out. Indeed, I am aware of only on substantial unanswered question arising from these five chapters (namely the existence of fini (pseudo-)colimits in $\mathfrak{Top}$, touched on in §4.2); doubtless there are many other minor points to b cleared up, and several theorems whose proofs will be improved and simplified in time, but th foundations of the subject do appear to be pretty stable. This is of course a bad thing: it is vital to th health of a subject as basic as topos theory that its fundamental tenets should be the subject continual review and improvement, and I am uncomfortably aware that by writing this book I hav contributed largely to the concreting-over of these foundations. My only defence against this charge that it seemed to me that the solidification was taking place anyway, and it was better that it shoul happen in print than in an unpublished folklore accessible only to insiders.

The average mathematician, who regards category theory as “generalized abstract nonsense”, tend to regard topos theory as generalized abstract category theory. (No doubt it has inherited th reputation from its parent, the Grothendieck approach to algebraic geometry.) And yet S. Mac Lan [179] regards the rise of topos theory as a symptom of the *decline* of abstraction in category theor and of abstract algebra in general. I am convinced that Mac Lane is right, and that his insight poin the way to the most probable future development of topos theory; for almost all the *recent* work significance in topos theory has been concerned not with toposes as an abstract and isolated area mathematics, but with toposes as an aid to understanding and clarifying concepts in other areas. (Se for example, [36], [57], [63], [79], [88], [90], [112], [130].)

To take a specific example, consider the general existence theorem for classifying toposes (6.56 One’s first reaction on seeing this theorem is to admire its elegance and generality; the secon reaction (which comes quite a long time later) is to realize its fundamental uselessness—a quali which, by the way, it shares with the General Adjoint Functor Theorem. For the only possible use o such a theorem is to reduce the study of a particular geometric theory to the study of its generic mod (or conversely, to reduce the study of a particular topos to that of the theory whose generic model contains), and the theorem as proved in §6.5 simply does not provide an effective means of passin from the one to the other. Thus the “syntactic” proof of the same theorem in §7.4, though appreciab messier, is much more valuable in practice—and it is this proof, not the later one given in the earli chapter, which has inspired most subsequent work on the subject.

In saying that the future of topos theory lies in the clarification of other areas of mathemati through the application of topos-theoretic ideas, I do not wish to imply that, like Grothendieck, I vie topos theory as a machine for the demolition of unsolved problems in algebraic geometry or anywhe else. On the contrary, I think it is unlikely that elementary topos theory itself will solve any maj outstanding problems of mathematics; but I do believe that the spreading of the topos-theoret outlook into many areas of mathematical activity will inevitably lead to the deeper understanding the real features of a problem which is an essential prelude to its correct solution.

What, then, is the topos-theoretic outlook? Briefly, it consists in the rejection of the idea that the is a fixed universe of “constant” sets within which mathematics can and should be developed, and t recognition that the notion of “variable structure” may be more conveniently handled within universe of *continuously variable* sets than by the method, traditional since the rise of abstract s theory, of considering separately a domain of variation (i.e. a topological space) and a succession constant structures attached to the points of this domain. In the words of F. W. Lawvere [LB], “Eve notion of constancy is relative, being derived perceptually or conceptually as a limiting case variation, and the undisputed value of such notions in clarifying variation is always limited by th origin. This applies in particular to the notion of constant set, and explains why so much of naïve s theory carries over in some form into the theory of variable sets”. It is this generalization of ide from constant to variable sets which lies at the heart of topos theory; and the reader who keeps it mind, as an ultimate objective, whilst reading this book, will gain a great deal of understandi thereby.

Next, a few words on some of the things which I have not done in this book.

(1) In the definition of a topos, I have taken cartesian closedness and the existence of Ω as tw separate axioms, instead of combining them into a single axiom of power-objects as suggested by Kock [66]. (The equivalence of Kock’s axiom is, however, covered in the Exercises to chapter 1.) At practical level, I would defend this decision on two grounds: (a) that there are a number of results the book (notably in chapter 2) which use only cartesian closedness and not the full topos axioms, a some (e.g. Theorem 1.47) where exponentials and Ω are used in essentially different ways in the sa proof; and (b) that if one takes the power-object definition, one is obliged (as in [WB]) to follow immediately with the rather technical proof that this definition implies cartesian closedness, and o is in danger of losing one’s readers at this critical point. On a more philosophical level, I would a (c) that the definition via power-objects is really a set-theorist’s rather than a category-theorist definition of a topos, in that it subordinates the notion of “function” to that of “subset” by means the set-theoretic device of identifying functions with their graphs. One of the principal features category theory is that it takes “morphism” as a primitive notion, on a level with (*not*, incidentall superior to) that of “object”; it is therefore right that the definition of a topos should include its clos structure.

(2) I have not introduced the Mitchell-Bénabou language until rather late in the book, at the end chapter 5.1. I know that there are some people whose ideal textbook on topos theory would begin wi the definition and just enough development of exactness properties to introduce the language a prove the soundness of its interpretation; thereafter all proofs would be conducted within the form language. I do not agree with this approach; I believe that it is impossible to appreciate the full pow of the Mitchell-Bénabou language until you have had some experience of proving things without (indeed, this is almost the only place in the book where I have consciously followed a particul ordering of material for pedagogical rather than logical reasons). There is also the point that t formal-language approach breaks down when confronted with the relative Giraud theorem (4.46 whilst the Mitchell-Bénabou language is a very powerful tool in proofs within a single topos, it is n well adapted to proofs in which we have to pass back and forth between two toposes by a geomet morphism. (It is possible that the proof of 4.46 could be shortened by using the language of local internal categories, but that is a different matter.)

(3) I have already mentioned that Freyd’s embedding theorem [FK] will not be found in this boo In consequence, Freyd’s concept of well-pointed topos plays a relatively minor role; it is n introduced until §9.3.

(4) I have not included any reference to Freyd's more recent development (unpublished as yet) o the ~~theory of *allegories*. This theory sets out to do for the category of sets and relations what~~ topo theory does for sets and functions; Freyd has been known to maintain that it provides a simpler an more natural basis than topos theory for many of the ideas developed in this book, but I personal remain unconvinced of this.

(5) I have not mentioned the work being done by D. Bourn [13], R. Street [113], [114] and other on the development of a 2-categorical analogue of topos theory. It appears to me, however, that th fundamentals of this theory have not yet reached a sufficiently definitive state for treatment in boo form.

(6) One generalization of topos theory whose omission I do slightly regreat is J. Penon's notion o *quasitopos* [99]. However, I feel that to introduce it early in the book would simply have introduce extra complications in the proofs without any benefits in the form of additional well-known example and to introduce it later on would have involved a good deal of duplication. I hope, nevertheless, th O. Wyler's forthcoming notes on quasitoposes (promised in [130]) will help to fill this gap.

(7) The phrase "Grothendieck universe" does not appear anywhere in the book. This is intentional; have deliberately been as vague as possible (except in §9.3) about the features of the set theory whic I am using, since it really doesn't matter. Topos theory is an elementary theory, and its main theorem are not—or ought not to be—dependent on recondite axioms of set theory. (In fact I am a fully paid up member of the Mathematicians' Liberation Movement founded by J. H. Conway [157].) If presse however, I would admit to using a Godel-Bernays-type set theory having a distinction between sma categories (sets) and large categories (proper classes); but I also wish to consider certain "very large 2-categories (notably $\mathfrak{Cat}$ and $\mathfrak{Top}$) whose objects are themselves large categories. If I wished to b strictly formal about this, I should need to introduce at least one Grothendieck universe; but since a the statements I wish to make about $\mathfrak{Cat}$ and $\mathfrak{Top}$ are (equivalent to) elementary ones, there is n *real* need to do so. In order to retain some set-theoretic respectability, I have limited myself t considering sheaves only on small sites; this has the disadvantage that we cannot state Giraud theorem in its slickest form (a category is a Grothendieck topos iff it is equivalent to the category canonical sheaves on itself), but is otherwise not as irksome as the authors of [GV] would have u believe.

Finally, I have to state my position on the most controversial question in the whole of topos theor how to spell the plural of topos. The reader will already have observed that I use the English plural; do so because (in its mathematical sense) the word topos is not a direct derivative of its Greek roo but a back-formation from topology. I have nothing further to say on the matter, except to ask thos toposophers† who persist in talking about topoi whether, when they go out for a ramble on a cold da they carry supplies of hot tea with them in thermoi.

† I am indebted to Miles Reid for suggesting the terms "toposopher" and "toposophy": and I urge my fellow-toposophers to ado them.

Notes for the Reader

Throughout the book, a single numbering system is used for definitions, lemmas, theorem remarks, etc.; the number *n* .*pq* normally denotes the *q*th numbered reference in the *p*th paragraph chapter *n*, except that in certain paragraphs which have more than nine numbered references, th numbers run on consecutively from *n*.*p*9. (Thus 8.20 is the tenth numbered reference in §8.1, and 5.5 the eleventh in §5.4.) Fortunately, it has been possible to do this without overlapping. While th system may not have logic on its side, it does combine the advantage of simple and easil remembered numbers (no profusion of decimal points!) with that of references which are easy locate. A. *n* denotes the *n*th numbered reference in the Appendix.

At the end of each chapter will be found a number of exercises: about ten on each of the earli chapters, more on the later ones. They vary considerably in difficulty, some being completely routin whereas others are quite substantial. I have not given any indication of which exercises I consider be easier (the order is that of the material in the preceding chapter to which they refer), but I hav given fairly copious hints for most of the harder ones. In quite a number of cases, the result of a exercise is used in either the exercises or the text of a subsequent chapter; these exercises a distinguished by an obelus (†).

The following summary of the logical interdependence of the various chapters may be useful to th reader who is interested in one particular topic. Chapter 0 contains a summary of certain backgrou material which is required either to motivate the definition of a topos, or to provide a source examples. Chapters 1–5 form the core of the book; of these chapters 1–4 follow a more or le geodesic path (with a few digressions such as §4.2) from the definition of a topos (1.11) to the relativ Giraud theorem (4.46) and the existence of pullbacks in $\mathfrak{BTop}/\mathscr{E}$ (4.48). The logical depende relation in these four chapters is thus fairly close to being a total order.

However, the majority of the material in chapter 2 (on internal categories) is fairly technical, a some readers may find it fairly difficult at a first reading. I would advise such readers to omit th whole of chapter 2, except for Theorem 2.32 (which is important, and has applications in other are than internal category theory), and go on to chapter 3. (There are some references to chapter 2 in §3. but you can refer back for these as necessary.) You may then go on to the first paragraph (only) chapter 4, the whole of chapter 5 except for some parts of §5.3, and even the first two paragraphs chapter 6, before returning to tackle chapter 2.

Chapter 5 introduces a number of concepts which, although part of the mainstream of topos theor are not involved in the proof of the relative Giraud theorem. In particular, it contains a description the internal language of a topos, which is used freely in the second half of the book.

The last four chapters present various extensions and applications of the basic theory; I original hoped to make them logically independent, so that they could be read in any order, but inevitab some cross-connections have established themselves. The following table summarizes the importa ones:

Before reading	*it is advisable to read*
7.4	6.3 and 6.5
8.1	7.5
8.4	6.2
9.1	6.2 and 6.4

There are some further cross-connections between the exercises on these four chapters (see, f example, exercises 6.11, 8.7, 9.6 and 9.14).

The Appendix is a presentation of material originally intended for inclusion in chapter 2; it w removed from there because, whilst it seems certain to become part of the mainstream of topos theo before very long, it is possible that the basic definition of “locally internal category” has not y reached its final form. It may be read at any time after chapter 2, although it does make a number references to later chapters.

Throughout the book, references to the bibliography are enclosed in square brackets. Th bibliography itself is divided into four sections: section A consists of “standard references” to oth areas of mathematics (e.g. lattice theory, algebraic topology) which are used when a theorem definition from one of these areas is quoted in the text. Section B contains works of a general natu on topos theory, and some introductory articles written for non-specialist readers (e.g. [MB], [WI] Section C contains the remaining references on topos theory, and a number of closely-related pape on category theory, sheaf theory, etc. Sections B and C together aim to present a complete list articles so far published on topos theory; however, I do not include short abstracts of talks, nor Ph. theses unless they contain important results not published elsewhere. Section D contains th remaining papers which are referred to in the text. References in sections A and B are indicated b two-letter codes; those in sections C and D are consecutively numbered. In all four sections, I hav indicated the *Mathematical Reviews* review numbers, where they exist.

TOPOS THEORY

Preliminaries

0.1. CATEGORY THEORY

This paragraph is not intended to give a detailed introduction to the basic ideas of category theor our intention is rather to indicate some of the concepts and theorems which we shall be assuming familiar, and to set out some standard notations. The reader who considers himself unfamiliar wi these basic concepts would be well advised to refer to the excellent book by Mac Lane [CW], or to ar other standard text on category theory, before proceeding further with this book.

We shall normally use script capitals ($\mathscr{C}$, $\mathscr{D}$, $\mathscr{E}$, . . .) to denote "large" categories. When we mal the assertion "$\mathscr{C}$ is a category", without further qualification, we mean that $\mathscr{C}$ is a model of tl "elementary theory of categories" [72], i.e. $\mathscr{C}$ is a *metacategory* in the sense of [CW], chapter I. Th means that we do not regard $\mathscr{C}$ as being formally defined within a particular model of set theory; particular, if X and Y are objects of $\mathscr{C}$, we do not require that the morphisms of $\mathscr{C}$ from X to Y sha form a set.

However, except in chapter 9, we shall normally assume that we are given a (fixed) model of son suitable set theory (including the axiom of choice when necessary); and we shall use the letter $\mathscr{S}$ denote the *category of sets and functions* which we obtain from it. We use the term *small category* f a category whose morphisms form a set. If C is a small category, we write $\mathscr{S}^{\mathbf{C}^{op}}$ for the category *presheaves* on C, i.e. contravariant functors from C to $\mathscr{S}$; amongst the objects of $\mathscr{S}^{\mathbf{C}^{op}}$, we have tl *representable* functors h_U, where U is an object of C, defined by $h_U(V) = \hom_{\mathrm{c}}(V, U)$. (F typographical reasons, we shall sometimes write $h(U)$ for h_U; we shall also write h^U for the covaria representable functor $\hom_{\mathrm{c}}(U, -)$.) We shall make frequent use of the following two results:

0.11 Lemma (Yoneda [187]). *For objects U, X of C and $\mathscr{S}^{\mathbf{C}^{op}}$ respectively, then is a bijection (natur in both variables) between morphisms $h_U \longrightarrow X$ in $\mathscr{S}^{\mathbf{C}^{op}}$ and elements of the set $X(U)$.* ■

0.12 Lemma. *Any object of $\mathscr{S}^{\mathbf{C}^{op}}$ can be expressed as a colimit of a diagram whose vertices a representable functors.*

Proof Let X be an object of $\mathscr{S}^{\mathbf{C}^{op}}$, and let $(\mathbf{C} \downarrow X)$ denote the (small) *comma category* whose objec are pairs (U, α), U an object of C and $\alpha : h_U \longrightarrow X$ in $\mathscr{S}^{\mathbf{C}^{op}}$, and whose morphism are commutati triangles

$$\begin{tikzcd} h_U \arrow[rr] \arrow[dr] & & h_V \arrow[dl] \\ & X & \end{tikzcd}$$

in $\mathscr{S}^{\mathbf{C}^{\mathrm{op}}}$.

~~Then we have an obvious "forgetful" diagram~~ $(\mathbf{C}\downarrow X) \longrightarrow \mathscr{S}^{\mathbf{C}^{\mathrm{op}}}$ ~~given by~~ $(U, \alpha) \longmapsto h_U$; ar the colimit of this diagram is readily seen to be X. ■

We assume that the reader is familiar with the notions of *limit* and *colimit*, and of *adjoint functor* (We use the notation $T \dashv G$) for "T is left adjoint to G".) A word of warning here: when we say that category *has limits* of a particular type, we mean that there is given, for each diagram of th appropriate type, a *canonical* choice of limit. Thus for example, when we say that $\mathscr{S}$ has bina products we mean that, given sets X and Y, there exists not merely a set whose elements are in 1- correspondence with ordered pairs $\langle x, y\rangle$, but a canonical such set, namely the set of all ordere pairs, itself. However, when we say that a functor *preserves limits*, we do not imply that it preserv the canonical choice of limits; and the statement that a given object is *a* limit of a certain diagra does not imply that it is the canonical one.

If a category $\mathscr{C}$ has a terminal object (i.e. a limit for the empty diagram), we denote it by 1; and if is any object of $\mathscr{C}$, we use X also to denote the unique morphism $X \longrightarrow 1$, and 1_X (or simply 1) f the identity morphism on X. (The confusion inherent in this notation may be loosely justified by th fact that 1_X is the terminal object of the category $\mathscr{C}/X$ of *objects over* X, and that if $Y \xrightarrow{f} X$ is an object of this category, f is also the unique morphism of $\mathscr{C}/X$ from f to 1_X.) If $\mathscr{C}$ has products and/ pullbacks, we use the letter π to denote the canonical projection of a product or pullback onto one its factors, with suffix 1, 2, 3, . . . to denote the first, second, third, . . . factor. Similarly, we sha usually (but not exclusively) use the letter v, with appropriate suffix, to denote the inclusion of on factor in a coproduct.

We also assume familiarity with the notions of *monad* (or triple) and *comonad*, and of *algebras* ov a monad. We shall make use of the "crude tripleability theorem" of Beck [153] in its "reflexiv coequalizer" form; recall that a parallel pair $X \underset{g}{\overset{f}{\rightrightarrows}}$ in a category $\mathscr{C}$ is said to be *reflexive* if the exists $Y \xrightarrow{h} X$ such that $fh = gh = 1_Y$. [In the case $\mathscr{C} = \mathscr{S}$, this is equivalent to saying that th image of $X \xrightarrow{(f,g)} Y \times Y$ is a reflexive binary relation on Y.]

0.13 Theorem. *Let* $\mathscr{C} \underset{U}{\overset{F}{\rightleftarrows}} \mathscr{A}$ *be a pair of functors with* $F \dashv U$, *and let* $\mathbb{H}$ *be the monad on* *induced by this adjunction. Suppose that* $\mathscr{A}$ *has coequalizers of reflexive pairs, that U preserves the and that U reflects isomorphisms. Then U is* monadic; *i.e. the comparison function* $K: \mathscr{A} \longrightarrow \mathscr{C}^{\mathbb{H}}$ *an equivalence of categories, where* $C^{\mathbb{H}}$ *denotes the category of* $\mathbb{H}$*-algebras.* ■

We shall also make use of the following theorems about categories of algebras:

0.14 Theorem (Eilenberg–Moore [162]). *Let* $\mathbb{H} = (H, \eta, \mu)$ *be a monad on* $\mathscr{C}$, *and suppose H has right adjoint G. Then there is a unique comonad structure* $\mathbb{G} = (G, \varepsilon, \delta)$ *on G, such that the categor* $\mathscr{C}_{\mathbb{G}}$ *of* $\mathbb{G}$*-coalgebras is isomorphic to* $\mathscr{C}^{\mathbb{H}}$, *by an isomorphism which identifies the two forget functors.* ■

0.15 Theorem ("Adjoint lifting theorem"; see [54]). *Let* $\mathbb{H}$, $\mathbb{K}$ *be monads on categories* $\mathscr{C}$, *respectively,* $T: \mathscr{C} \longrightarrow \mathscr{D}$ *a functor, and* $\overline{T}: \mathscr{C}^{\mathbb{H}} \longrightarrow \mathscr{D}^{\mathbb{K}}$ *a functor which is a "lifting" of T in t* *sense that the square*

$$\begin{tikzcd} \mathscr{C}^{\mathbb{H}} \arrow[r, "\overline{T}"] \arrow[d, "U"] & \mathscr{D}^{\mathbb{K}} \arrow[d, "U"] \\ \mathscr{C} \arrow[r, "T"] & \mathscr{D} \end{tikzcd}$$

commutes, where the U's denote forgetful functors. Suppose also that $\mathscr{C}^{\mathbb{H}}$ *has coequalizers of reflexi pairs. Then if T has a left adjoint, so has* $\overline{T}$. ■

0.16 Theorem (Linton [178]). *Let* $\mathbb{H}$ *be a monad on a category* $\mathscr{C}$, *and suppose that* $\mathscr{C}^{\mathbb{H}}$ *h coequalizers of reflexive pairs. Then if* $\mathscr{C}$ *has all finite* (*resp. all* $\mathscr{S}$*-indexed*) *coproducts,* $\mathscr{C}^{\mathbb{H}}$ *has a finite* (*resp. all* $\mathscr{S}$*-indexed*) *colimits.* ■

It is clear from 0.13, 0.15 and 0.16 that coequalizers of reflexive pairs ("reflexive coequalizers play an important rôle in the theory of monads. It is therefore appropriate to insert at this point lemma which, although capable of very wide application (as we shall see in chapter 6), has not foun its way into the standard texts on category theory.

0.17 Lemma. *Let*

$$\begin{tikzcd} X_1 \arrow[r, shift left, "f_1"] \arrow[r, shift right, "f_2"'] \arrow[d, shift right, "\alpha_1"'] \arrow[d, shift left, "\alpha_2"] & X_2 \arrow[r, "f_3"] \arrow[d, shift right, "\beta_1"'] \arrow[d, shift left, "\beta_2"] & X_3 \arrow[d, shift right, "\gamma_1"'] \arrow[d, shift left, "\gamma_2"] \\ Y_1 \arrow[r, shift left, "g_1"] \arrow[r, shift right, "g_2"'] \arrow[d, "\alpha_3"] & Y_2 \arrow[r, "g_3"] \arrow[d, "\beta_3"] & Y_3 \arrow[d, "\gamma_3"] \\ Z_1 \arrow[r, shift left, "h_1"] \arrow[r, shift right, "h_2"'] & Z_2 \arrow[r, "h_3"] & Z_3 \end{tikzcd}$$

be a diagram in any category satisfying the "obvious" commutativity conditions (*i.e.* $\beta_i f_j = g_j \alpha_i$ *for i* 1, 2, *j* = 1, 2, *etc.*), *in which the rows and columns are coequalizers and the pairs* (f_1, f_2) *and* $(\alpha_1, \alpha$ *are reflexive. Then the diagonal* $X_1 \overset{\beta_1 f_1}{\underset{\beta_2 f_2}{\rightrightarrows}} Y_2 \xrightarrow{\gamma_3 g_3} Z_3$ *is a coequalizer.*

Proof. First we note that

$$\begin{aligned} \gamma_3 &= \operatorname{coeq}(\gamma_1, \gamma_2) \\ &= \operatorname{coeq}(\gamma_1 f_3, \gamma_2 f_3) \text{ since } f_3 \text{ is epi} \\ &= \operatorname{coeq}(g_3 \beta_1, g_3 \beta_2). \end{aligned}$$

So the lower right-hand square is a pushout; i.e. a morphism $Y_2 \xrightarrow{\theta} T$ factors throug $\overline{Y_2 \xrightarrow{\gamma_3 g_3} Z_3}$ iff it coequalizes both (g_1, g_2) and (β_1, β_2). But if this condition is satisfied, then $\theta\beta_1$ $= \theta\beta_2 f_1 = \theta g_1\alpha_2 = \theta g_2\alpha_2 = \theta\beta_2 f_2$. Conversely, if $\theta\beta_1 f_1 = \theta\beta_2 f_2$ and $X_2 \xrightarrow{s} X_1$ is a common splittin for f_1 and f_2, then $\theta\beta_1 = \theta\beta_1 f_1 s = \theta\beta_2 f_2 s = \theta\beta_2$; and similarly $\theta g_1 = \theta g_2$. So $Y_2 \longrightarrow Z_3$ is a coequaliz of $\beta_1 f_1$ and $\beta_2 f_2$. ■

In conclusion, we must mention two areas of category theory not covered in [CW]. One is th theory of 2-categories: a 2-category $\mathfrak{C}$ is a category whose "hom-sets" have the structure of (n necessarily small) categories. That is to say for each pair of objects (X, Y) of $\mathfrak{C}$, we have a category (X, Y) whose objects are the morphisms (or *1-arrows*) of $\mathfrak{C}$ from X to Y, and whose morphisms a called *2-arrows* of $\mathfrak{C}$; and the operation of composing 1-arrows of $\mathfrak{C}$ is functorial in both variable Introductions to the theory of 2-categories will be found in [155], [167] and [172].

The reader is warned that, when discussing 2-categories, we shall normally use terms such "functor" and "limit", and notations such as $\mathfrak{C}/X$, in what the Australian school [172] would call th "pseudo" sense, in which diagrams which commute "on the nose" are replaced by diagram commuting up to a (specified) 2-isomorphism. For example, $\mathfrak{C}/X$ is the 2-category whose objects a 1-arrows $Y \xrightarrow{f} X$ of $\mathfrak{C}$, and whose 1-arrows are triangles

$$\begin{tikzcd}
Y_1 \arrow[rr, "g"] \arrow[dr, "f_1"'] \arrow[rr, phantom, shift right=4ex, "\nearrow\alpha"] & & Y_2 \arrow[dl, "f_2"] \\
& X &
\end{tikzcd}$$

commuting up to a specified 2-isomorphism α; a 2-arrow of $\mathfrak{C}/X$ from (g_1, α_1) to (g_2, α_2) is a 2-arro $g_1 \xrightarrow{\beta} g_2$ of $\mathfrak{C}$ such that $(f_2 * \beta)\alpha_1 = \alpha_2$. Similarly, when we speak of a functor $T: \mathfrak{C} \longrightarrow$ between 2-categories, we do not imply that T commutes exactly with composition of 1-arrows, b only up to coherent natural 2-isomorphism. (It is understood that if $\mathscr{C}$ is an ordinary category, v identify it with the *locally discrete* 2-category whose objects and 1-arrows are the objects ar morphisms of $\mathscr{C}$, and whose only 2-arrows are identities.)

Our reason for departing from the Australian convention on nomenclature is that it is the "pseudo" concepts which arise most frequently in practice. If we wish to emphasize the fact that particular functor commutes exactly with composition of 1-arrows (i.e. is a functor in the Australia sense), we shall call it a *strict functor*. In one paragraph (4.2), we shall have occasion to consider th still weaker notion of *lax functor*, in which the 2-arrows up to which diagrams commute need not eve be invertible; we shall define these lax concepts explicitly when we need them. And in §2.4 we sha encounter an example of a *bicategory*, which is a "pseudo-2-category" in the Australian sense; i.e. it defined by the same data as a 2-category, but the unitary and associative laws for composition of arrows hold only up to coherent natural 2-isomorphism.

Finally, we must introduce the notion of category of fractions. If $\mathscr{C}$ is a category and Σ a class morphisms of $\mathscr{C}$, then the *category of fractions* $\mathscr{C}\Sigma^{-1}$ is defined (up to equivalence) by requiring th there exist a functor $P_\Sigma: \mathscr{C} \longrightarrow \mathscr{C}\Sigma^{-1}$ which is universal among functors $\mathscr{C} \longrightarrow \mathscr{D}$ sending all th morpnisms in Σ to isomorphisms.

0.18 DEFINITION. A class Σ of morphisms of $\mathscr{C}$ is said to admit a *calculus of right fractions* if:

(i) Σ is closed under composition, and contains all identity morphisms of $\mathscr{C}$.

(ii) Given a diagram

$$\begin{tikzcd} X \arrow[dr, "f"'] & & Y \arrow[dl, "\sigma"] \\ & Z & \end{tikzcd} \quad \text{with } \sigma \in \Sigma \text{, there exists} \quad \begin{tikzcd} & T \arrow[dl, "\tau"'] \arrow[dr, "g"] & \\ X & & Y \end{tikzcd}$$

with $\tau \in \Sigma$, such that $f\tau = \sigma g$. (Note: T need not be a pullback.)

(iii) Given a diagram $X \overset{f}{\underset{g}{\rightrightarrows}} Y \xrightarrow{\sigma} Z$ with $\sigma \in \Sigma$, $\sigma f = \sigma g$, there exists $T \xrightarrow{\tau} X$ in Σ with $f\tau = g\tau$. ■

0.19 THEOREM (Gabriel–Zisman [CF]). *Let $\mathscr{C}$ be a category with finite limits, and Σ a class of morphisms of $\mathscr{C}$ admitting a calculus of right fractions. Then the category of fractions $\mathscr{C}\Sigma^{-1}$ has finite limits, and P_Σ preserves them.*

Proof. We show that $\mathscr{C}\Sigma^{-1}$ can be described as follows: its objects are those of $\mathscr{C}$, and morphisms $X \longrightarrow Y$ in $\mathscr{C}\Sigma^{-1}$ are equivalence classes of diagrams

$$\begin{tikzcd} Z \arrow[r, "f"] \arrow[d, "\sigma"] & Y \\ X & \end{tikzcd}$$

in $\mathscr{C}(\sigma \in \Sigma)$, under the relation that (σ_1, f_1) is equivalent to (σ_2, f_2) iff there exists

$$\begin{tikzcd} & T \arrow[dl, "\tau_1"'] \arrow[dr, "\tau_2"] & \\ Z_1 & & Z_2 \end{tikzcd}$$

such that $\sigma_1\tau_2 = \sigma_2\tau_2 \in \Sigma$ and $f_1\tau_2 = f_2\tau_2$. (It is not hard to verify that this does define an equivalence relation.) To compose two morphisms

$$\begin{tikzcd}[column sep=small] & T \arrow[dl, "\sigma_1"'] \arrow[dr, "f_3"] & & U \arrow[dl, "\sigma_2"'] \arrow[dr, "f_2"] & \\ X & & Y & & Z \end{tikzcd}$$

in $\mathscr{C}\Sigma^{-1}$, we form a commutative square

$$\begin{tikzcd} V \arrow[r, "g"] \arrow[d, "\tau"] & U \arrow[d, "\sigma_2"] \\ T \arrow[r, "f_1"] & Y \end{tikzcd}$$

with $\tau \in \Sigma$ (using 0·18(ii)), and then define $(\sigma_2, f_2)(\sigma_1, f_1) = (\sigma_1 \tau, f_2 g)$. Again it is easy to check th this definition is independent of the choice of representatives (σ_i, f_i) and of the choice of V; hence als the composition it defines is associative. The functor P_Σ is defined by

$$P_\Sigma(X) = X,\ P_\Sigma(X \xrightarrow{f} Y) = \begin{tikzcd} X \arrow[r, "f"] \arrow[d, "1"] & Y \\ X & \end{tikzcd} ;$$

its universal property is immediate. We now show that $\mathscr{C}\Sigma^{-1}$ has equalizers and P_Σ preserves ther (The argument for finite products is similar but easier.) Let

$$\begin{tikzcd} Z_i \arrow[r, "f_i"] \arrow[d, "\sigma_i"] & Y \quad (i = 1, 2) \\ X & \end{tikzcd}$$

be a parallel pair of morphisms of $\mathscr{C}\Sigma^{-1}$; then by a suitable choice of representatives we may ensu that $Z_1 = Z_2$ and $\sigma_1 = \sigma_2$. Let $E \xrightarrow{q} Z$ be the equalizer of f_1 and f_2 in C. Now let

$$\begin{tikzcd} U \arrow[r, "g"] \arrow[d, "\tau"] & X \\ T & \end{tikzcd}$$

be a morphism of $\mathscr{C}\Sigma^{-1}$ equalizing (σ, f_1) and (σ, f_2); then for a suitable choice of commutati square

$$\begin{tikzcd} V \arrow[r, "h"] \arrow[d, "\rho"] & Z \arrow[d, "\sigma"] \\ U \arrow[r, "g"] & X \end{tikzcd}$$

with $\tau p \in \sum$ we can ensure that $f_1 h = f_2 h$ in C, and so there exists a factorization $V \xrightarrow{k} E$ of through q. Now

$$\begin{tikzcd} V \arrow[r, "k"] \arrow[d, "\tau\rho"] & E \\ T & \end{tikzcd}$$

is a factorization of (τ, g) through $P_\Sigma(\sigma q)$ in $\mathscr{C}\Sigma^{-1}$; and it is not hard to verify that it is unique. S $P_\Sigma(\sigma q)$ is an equalizer of (σ, f_1) and (σ, f_2) in $\mathscr{C}\Sigma^{-1}$; hence also P_Σ preserves equalizers. ■

0.2. SHEAF THEORY

As in the previous paragraph, we do not intend to give a detailed account of the “classical” theory sheaves on a topological space; for this the reader is referred to one of the standard texts on th subject, for example [ST] or [TF]. However, a brief review of the classical theory is indispensable an aid to understanding the motivation for the more general theory of Grothendieck topologies whic follows in the next paragraph.

Let $(X, \mathbf{T})$ be a topological space. The set $\mathbf{T}$ of open subsets of X is partially ordered by inclusion, s we can regard it as a small category in the usual way; i.e. the objects of $\mathbf{T}$ are the open sets, and $\mathbf{T}$ h one morphism from U to V iff $U \subseteq V$.

0.21 Definition. A *presheaf* (of sets) on X is a presheaf on the category $\mathbf{T}$, i.e. a contravariant funct from $\mathbf{T}$ to $\mathscr{S}$. A presheaf P is thus specified by giving a set $P(U)$ for each open $U \subseteq X$, and *restriction map*

$$\rho^U_V : P(U) \longrightarrow P(V)$$

for each $V \subseteq U$, subject to the obvious compatibility conditions. A *morphism of presheaves* is just natural transformation of functors. ■

The notion of presheaf includes many examples which are familiar concepts in elementary topology.

0.22 Examples. (i) For any set A, we have the *constant presheaf* $\overline{A}$ defined by $\overline{A}(U) = A$, $\rho^U_V = 1$ for all $V \subseteq U \subseteq X$.

(ii) For any open $U \subseteq X$, we have the representable presheaf h_U, defined by