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Trigonometry

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Stan Gibilisco

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To Tim, Samuel, and Tony

About the Author

Stan Gibilisco, an electronics engineer, researcher, and mathematician, has authored multiple titles for the McGraw-Hill *Demystified* and *Know-It-All* series, along with numerous other technical books and dozens of magazine articles. His work has been published in several languages.

Contents

Introduction

This book can help you learn the principles of trigonometry without taking a formal course. It can also serve as a supplemental text in a classroom, tutored, or home-schooling environment. None of the mathematics goes beyond the high-school-senior level (12th grade). If you need a "refresher," yo can select from several *Demystified* books dedicated to mathematics topics. In particular, I recommend *Geometry Demystified* as a prerequisite for this course. If you want to build a "rock-solid" mathematics foundation before you start this course, you can go through *Pre-Algebra Demystified*, *Algebra Demystified*, *Geometry Demystified*, and *Algebra Know-It-All*. If you want to extend your knowledge beyond the scope of this course, I recommend *Pre-Calculus Know-It-All* and *Calculus Know-It-All*.

How to Use This Book

This book contains abundant multiple-choice questions written in standardized test format. You'll find an "open-book" quiz at the end of every chapter. You may (and should) refer to the chapter texts when taking these quizzes. Write down your answers, and then give your list of answers to a friend. Have your friend reveal your score, but not which questions you missed. The correct answer choices appear in the back of the book. Stick with a chapter until you get most of the quiz answers correct.

Two major parts constitute this course. Each part ends with a multiple-choice test. Take these test when you're done with the respective parts and have taken all the chapter quizzes. Don't look back a the text when taking the part tests. They're easier than the chapter-ending quizzes, and they don't require you to memorize trivial things. A satisfactory score is three-quarters correct. The correct answer choices appear in the back of the book.

The course concludes with a 100-question final exam. Take it when you've finished all the parts, all the part tests, and all of the chapter quizzes. A satisfactory score is at least 75 percent correct answers. Again, the correct answer choices are listed in the back of the book.

With the part tests and the final exam, as with the quizzes, have a friend divulge your score without letting you know which questions you missed. That way, you won't subconsciously memoriz the answers. You might want to take each test, and the final exam, two or three times. When you get score that makes you happy, you can (and should) check to see where your strengths and weaknesses lie.

I've posted explanations for the chapter-quiz answers (but not for the part-test or final-exam answers) on the Internet. As we all know, Internet particulars change; but if you conduct a phrase search on "Stan Gibilisco," you'll get my Web site as one of the first hits. You'll find a link to the explanations on that site. As of this writing, it's **www.sciencewriter.net**.

Strive to complete one chapter of this book every 10 days or two weeks. Don't rush, but don't go too slowly either. Proceed at a steady pace and keep it up. That way, you'll complete the course in a few months. (As much as we all wish otherwise, nothing can substitute for "good study habits.") When you've finished up, you can use this book as a permanent reference.

I welcome your ideas and suggestions for future editions.

Stan Gibilisc

Part I
Origins and Theory

chapter 1
Angles, Distances, and Triangles

The word *trigonometry* comes from the Sanskrit expression “triangle measurement.” In its simplest form, trigonometry involves the relationships between the sides and angles of triangles where one of the angles measures 90° (a *right angle*). But, as you’ll learn as you go through this book, trigonometr involves a lot more than triangles! Before we get into “real trigonometry,” let’s review the basic principles and jargon for angles, distances, and triangles on *Euclidean* (flat) surfaces.

CHAPTER OBJECTIVES

In this chapter, you will

- Quantify angles in radians and degrees.
- Learn how lines, angles, and distances relate on flat surfaces.
- Classify triangles according to their interior angles.
- Define the trigonometric functions as ratios among lengths of the sides of a right triangle.
- Learn the Pythagorean theorem for right triangles.

Angles and Distances

When two lines intersect, we get four distinct angles at the point of intersection. In most cases, we’ll find that two of the angles are “sharp” and two are “dull.” If all four of the angles happen to be identical, then they all constitute right angles, and we say that the lines run *perpendicular, orthogonal,* or *normal* to each other at the intersection point. We can also define an angle using three points connected by two line segments. In that case, the angle appears between the line segments.

Naming Angles

The italic, lowercase Greek letter *theta* has become popular as an “angle name.” It looks like an italic numeral *0* with a horizontal line through it (θ). When writing about two different angles, a second Greek letter can go along with θ. The italic, lowercase letter *phi* works well for this purpose. It looks like an italic lowercase English letter *o* with a forward slash through it (ϕ).

Sometimes, mathematicians use the italic, lowercase Greek letters *alpha*, *beta*, and *gamma* to represent angles. We write these symbols as α, β, and γ, respectively. When things get messy and we have a lot of angles to write about, we can add numeric subscripts to italic Greek letters and denote our angles as symbols such as θ_1, θ_2, θ_3, and so on, or α_1, α_2, α_3, and so on.

TIP ***If you don’t like Greek symbols, you can represent angle variables with more familiar characters such as x, y, and z. As long as you know the context and stay consistent in a given situation, you can call an angle anything you want.***

Radian Measure

Imagine two *rays* (or “half-lines”) pointing out from the center of a circle so that each ray intersects the ~~circle at a specific point. Suppose that the distance between these points, as measured along~~ the curve of the circle (not along a straight line), is the same as the circle’s radius. In that case, the angle between the rays measures one *radian* (1 rad).

A circle has 2π rad going exactly once around, where π (the lowercase Greek letter pi, not in italics) stands for the ratio of a circle’s circumference to its diameter. This number is the same for all perfect circles in a flat plane. The number π is *irrational*, meaning that we can’t express it as a ratio between whole numbers. It equals approximately 3.14159.

Mathematicians commonly use the radian as their standard unit of angular measure. Sometimes, they’ll omit the “rad” after an angle when they know that they’re working with radians (and when they’re sure that you know it too). Based on that convention, we can sum up the nature of the radian as follows:

- An angle of π/2 represents 1/4 of a circle
- An angle of π represents 1/2 of a circle
- An angle of 3π/2 represents 3/4 of a circle
- An angle of 2π represents a full circle

We can categorize angles into several types:

- An *acute angle* has a measure of more than 0 but less than π/2
- A *right angle* has a measure of exactly π/2
- An *obtuse angle* has a measure of more than π/2 but less than π
- A *straight angle* has a measure of exactly π
- A *reflex angle* has a measure of more than π but less than 2π

Degree Measure

The *angular degree* (°), also called the *degree of arc*, is the unit of angular measure familiar to lay people. One degree (1°) represents 1/360 of a full circle. We can summarize degree measure by noting a few facts:

- An angle of 90° represents 1/4 of a circle
- An angle of 180° represents 1/2 of a circle
- An angle of 270° represents 3/4 of a circle
- An angle of 360° represents a full circle

When we use degrees, the general angle types break down as follows:

- An acute angle has a measure of more than 0 but less than 90°
- A right angle has a measure of exactly 90°
- An obtuse angle has a measure of more than 90° but less than 180°
- A straight angle has a measure of exactly 180°
- A reflex angle has a measure of more than 180° but less than 360°

TIP ***Whenever you see a quantitative (numerical) reference to an angle and no unit goes with it, ~~and especially if you see the symbol π in the expression, you can assume that the author wants~~ to express the angle in radians. However, you should always check the context to make certain! If you're writing a paper and you want to express an angle in radians, you can write "rad" after the quantity for the angle value. That way, you can make sure that your readers won't get confused. If you write "2 rad," for example, your readers will know that you mean two radians, not two degrees. Similarly, if you want your reader to know for sure that you mean to express an angle in degrees, you should put a little degree symbol after the quantity.***

Minutes and Seconds of Arc

Once in awhile, you'll read about fractions of a degree called *minutes of arc* or *arc minutes*. One arc minute (symbolized as 1 arc min or 1′) equals 1/60 of a degree. You might also read about *seconds of arc*, also known as *arc seconds*. One arc second (symbolized as 1 arc sec or 1″) equals 1/60 of an arc minute, or 1/3600 of a degree.

TIP ***Arc minutes and arc seconds differ from the minutes and seconds that astronomers sometimes use for defining the positions of objects in the sky. You'll learn about those minutes and seconds in Chap. 7.***

PROBLEM 1-1

Imagine that the measure of a certain angle *θ* equals π/6. What fraction of a complete circular rotation does this angle represent? What's the measure of *θ* in degrees?

SOLUTION

A full circular rotation represents an angle of 2π. The value π/6 equals 1/12 of 2π. Therefore, the angle *θ* represents 1/12 of a full circle. In degree measure, that's 1/12 of 360°, or 30°.

Still Struggling

If you've always measured and defined angles in degrees, the radian can seem strange at first. "Why," you ask, "would anyone want to divide a circle into an *irrational* number of angular parts?" Mathematicians prefer the radian-measure system because it works out more simply (believe it or not) than the degree-measure scheme in some disciplines. The radian is actually a *more natural* angular quantity than the degree! We can define the radian in geometric terms without talking about any numbers whatsoever, just as we can define the diagonal of a square as the distance from one corner to the opposite corner. The radian is a purely geometric unit. The degree was contrived by humans, probably dating back to ancient times when people knew that a year contained roughly 360 days. When we think of a year as a "circle in time," it's easy to extend the idea to all geometric circles. If our distant ancestors had seriously considered the true length of the year, which comes out to a fractional number of days (and, one might argue, an irrational number that doesn't even stay the same as the centuries pass), maybe they'd have chosen some other standard angle to represent the degree, such as 1/100 of a circle or 1/1000 of a circle. Even so, the radian is the most natural unit possible!

Angle Notation

Imagine that *P*, *Q*, and *R* represent three distinct points. Let *L* represent the line segment connecting and *Q*, and let *M* represent the line segment connecting *R* and *Q*. We can denote the angle between *L* and *M*, as measured at point *Q* in the plane defined by the three points, by writing $\angle PQR$ or *ARQP* as shown in Fig. 1-1.

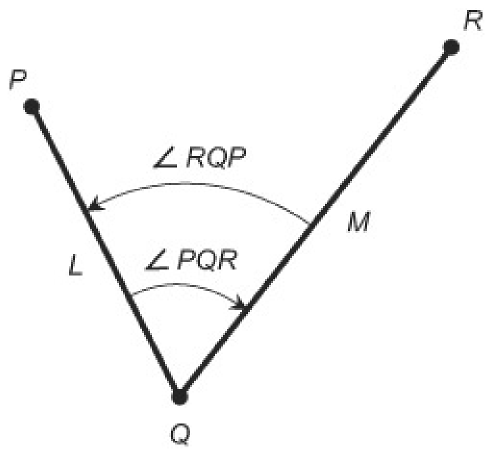

FIGURE 1-1. Angle notation.

If we want to specify the *rotational sense* of the angle, then $\angle RQP$ indicates the angle as we turn counterclockwise from *M* to *L*, and $\angle PQR$ indicates the angle as we turn clockwise from *L* to *M*. We consider counterclockwise-going angles as having positive values, and clockwise-going angles as having negative values.

In the situation of Fig. 1-1, $\angle RQP$ is positive while $\angle PQR$ is negative. If we make an approximate guess as to the measures of the angles in Fig. 1-1, we might say that $\angle RQP = +60°$ while $\angle PQR = -60°$.

TIP ***Rotational sense doesn't make any difference in basic geometry. However, it does matter when we work in* coordinate geometry *involving graphs. We'll get into coordinate geometry, also known as* analytic geometry*, later in this book. For now, let's not worry about the rotational sense in which we express or measure an angle. We can consider all angles as having positive measures.***

Angle Bisection Principle

Let's look at an angle called $\angle PQR$ that measures less than 180°, and that we can define with three points *P*, *Q*, and *R* as shown in Fig. 1-2. In this situation, there's *exactly one* (in other words, *one and only one*) ray *M* that *bisects* $\angle PQR$ (divides $\angle PQR$ in half). If *S* represents a point on *M* other than point *Q*, then $\angle PQS = \angle SQR$.

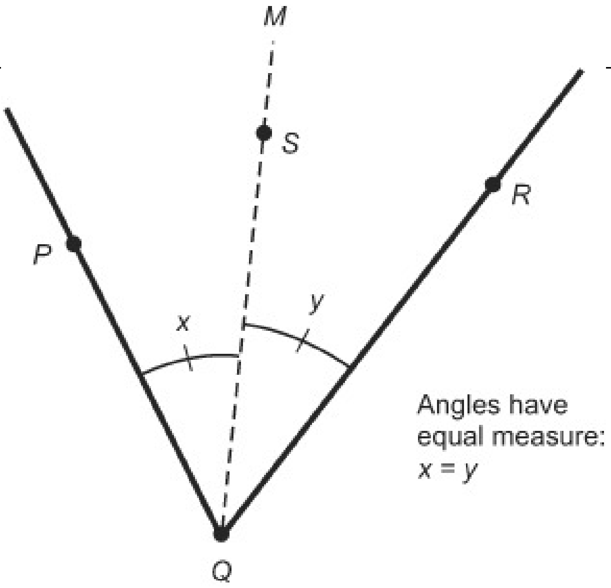

FIGURE 1-2. The angle bisection principle.

Perpendicular Principle

Consider a line *L* that passes through two distinct points *P* and *Q* as shown in Fig. 1-3. Suppose that *R* represents a point that doesn't lie on *L*. In that case, there's exactly one line *M* through point *R*, intersecting line *L* at some point *S*, such that *M* runs perpendicular to *L* (*M* and *L* intersect at a right angle) at point *S*.

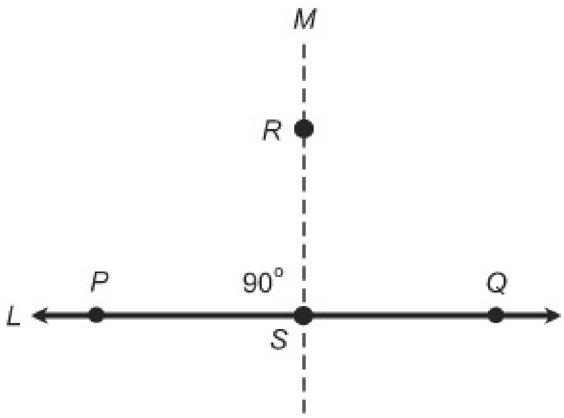

FIGURE 1-3 The perpendicular principle.

Perpendicular Bisector Principle

When we have a line segment connecting two points *P* and *R*, we can always find exactly one line *M* that runs perpendicular to line segment *PR* and that intersects *PR* at a point *Q*, such that the distance from *P* to *Q* (which we can write as *PQ*) equals the distance from *Q* to *R* (which we denote as *QR*). In other words, every line segment has exactly one *perpendicular bisector*. Figure 1-4 illustrates this principle.

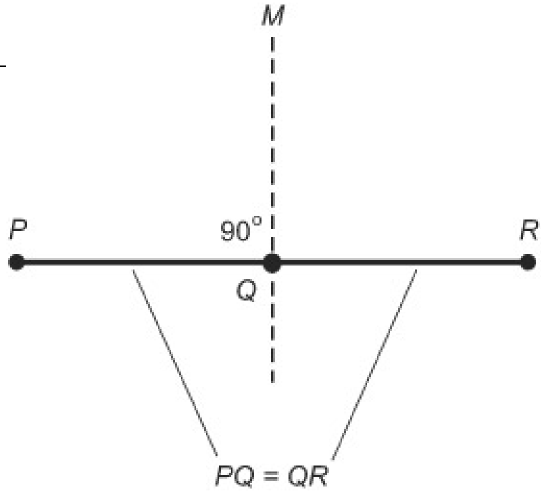

FIGURE 1-4. The perpendicular bisector principle. Illustration for Problem 1-2.

Distance Addition and Subtraction

Imagine that *P*, *Q*, and *R* represent points on a line, arranged so that *Q* lies somewhere between *P* and *R*. The following equations hold true concerning the distances between pairs of points as we measure them along the line as shown in Fig. 1-5:

$$PQ + QR = PR$$

$$PR - PQ = QR$$

$$PR - QR = PQ$$

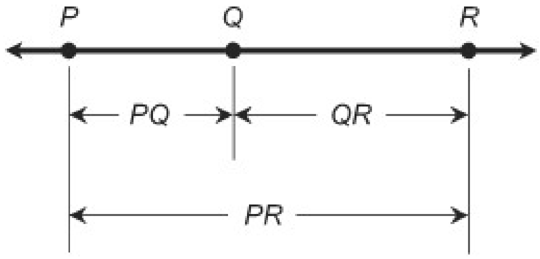

FIGURE 1-5. Distance addition and subtraction.

Angular Addition and Subtraction

Suppose that *P*, *Q*, *R*, and *S* represent points that all lie in the same *Euclidean plane*. In other words, all four points lie on a single, perfectly flat surface. Let *Q* represent the convergence point of three angles $\angle PQR$, $\angle PQS$, and $\angle SQR$, with line segment *QS* between line segments *QP* and *QR* as shown in Fig. 1-6. In that case, we'll always find that the following equations hold true:

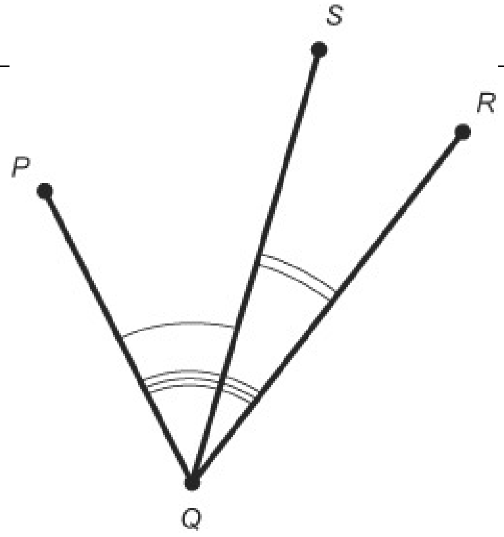

FIGURE 1-6. Angular addition and subtraction. Illustration for Problem 1-3.

$$\angle PQS + \angle SQR = \angle PQR$$
$$\angle PQR - \angle PQS = \angle SQR$$
$$\angle PQR - \angle SQR = \angle PQS$$

PROBLEM 1-2

Go back to Fig. 1-4 and examine it again. Imagine some point *S*, other than point *Q*, that lies on line *M* (the perpendicular bisector of line segment *PR*). What can you say about the lengths of line segments *PS* and *SR?*

SOLUTION

You can "streamline" the solutions to problems like this by making your own drawings. With the aid of your own sketch, you should see that for any point *S* that lies on line *M* (other than point *Q*), the distance *PS* exceeds the distance *PQ* (that is, $PS > PQ$), and the distance *SR* exceeds the distance *QR* (that is, $SR >$ QR).

PROBLEM 1-3

Look again at Fig. 1-6. Suppose that you move point *S* either straight toward yourself (out of th page) or straight away from yourself (back behind the page). In either case, point S no longer lies in the same plane as points *P*, *Q*, and *R* do. What can you say about the measures of $\angle PQR$, $\angle$ *PQS*, and $\angle SQR$?

SOLUTION

You can use your "three-dimensional mind's eye" to envision these situations. Either wa you should be able to see that the sum of the measures of $\angle PQS$ and $\angle SQR$ exceeds the measure of $\angle PQR$ because the measures of $\angle PQS$ and $\angle SQR$ both increase if point *S* departs *at a right angle* from the plane containing points *P*, *Q*, and *R*.

Triangles

In technical terms, a *triangle* consists of three line segments, joined pairwise at their end points, and including those end points. In order to determine a triangle, the three points must not be *collinear* (they must not all lie on the same straight line). For now, let's assume that the surface or *universe* fo all our triangles is Euclidean ("flat"), and not "curved" like the surface of a sphere, cone, or cylinder

TIP ***In a Euclidean universe, we can always determine the shortest distance between two points by finding the straight line segment connecting the points, and then measuring the length of the line segment. Conversely, if the shortest distance between two points always constitutes a straight line, then we know that we're working in a Euclidean universe.***

Vertices, Names, and Sides

Figure 1-7 shows three points called *A*, *B*, and *C*, connected by line segments to form a triangle. We call each point a *vertex* of the triangle, so the figure has three *vertices* (or *vertexes*). We call the who figure "triangle *ABC*."

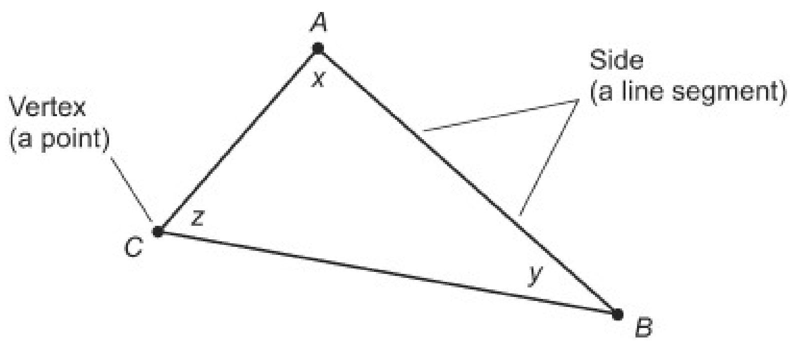

FIGURE 1-7. Vertices, sides, and angles of a triangle.

When naming triangles, geometers write an uppercase Greek letter delta (Δ) in place of the word "triangle." According to that notation, Fig. 1-7 portrays Δ*ABC*. Alternatively, we can call it Δ*BCA*, Δ*CAB*, Δ*CBA*, Δ*BAC*, or Δ*ACB*.

We can name the sides of the triangle in Fig. 1-7 according to their end points. In this case, our triangle has three sides: line segment *AB* (or *BA*), line segment *BC* (or *CB*), and line segment *CA* (or *AC*).

Interior Angles

Each vertex of a triangle corresponds to a specific *interior angle*, which always measures more than 0° (or 0 if we express it in radians) but less than 180° (or π if we express it in radians). In Fig. 1-7, w denote the interior angles using the lowercase italic English letters *x*, *y*, and *z*. Alternatively, we can use italic lowercase Greek letters to symbolize the interior angles. Subscripts can help us distinguish the angles from one another; for example, θ_a, θ_b, and θ_c could represent the interior angles at vertice *A*, *B*, and *C*, respectively.

Directly Similar Triangles

Two triangles are *directly similar* if and only if they have the same proportions in the same rotationa sense (that is, as we go around them both in the same direction). Two triangles *are not* directly ~~similar if we must flip one of them over, in addition to changing its size and rotating it, in order~~ to place it exactly over the other one.

Directly Congruent Triangles

Two triangles are *directly congruent* if and only if they're directly similar and the corresponding sides have identical lengths. If we take one of the triangles and rotate it clockwise or counterclockwise to the correct extent, we can "paste" it precisely over the other one. Rotation and motion are allowed, but flipping-over, also called *mirroring*, is forbidden.

Still Struggling

You should remember these fundamental facts when you work with triangles that look a lot alike:

- If two or more triangles are directly congruent, then the corresponding sides have equal lengths as you proceed around them all in the same direction. The converse also holds true. If two or more triangles have corresponding sides with equal lengths as you proceed around them all in the same direction, then all the triangles are directly congruent.
- If two or more triangles are directly congruent, then the corresponding interior angles have equal measures as you proceed around them all in the same direction. However, the converse does not always hold true. Two or more triangles can have corresponding interior angles with equal measures when you go around them all in the same direction, and nevertheless fail to be directly congruent (although they'll always be directly similar).

Acute Triangle

We have an *acute triangle* if and only if each of the three interior angles is acute. In such a triangle, none of the angles measure as much as a right angle (90° or $\pi/2$); they're all smaller than that. Figure 1-8 shows some examples.

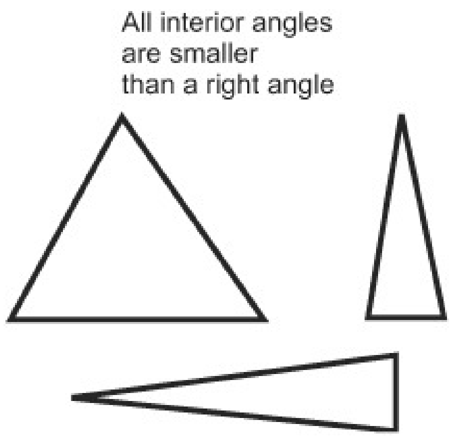

FIGURE 1-8 In an acute triangle, all angles measure less than a right angle.

Obtuse Triangle

We have an *obtuse triangle* if and only if one of the three interior angles is obtuse, measuring more than a right angle (90° or π/2) but less than a straight angle (180° or π). In a triangle of this type, the two nonobtuse angles are both acute. Figure 1-9 shows some examples.

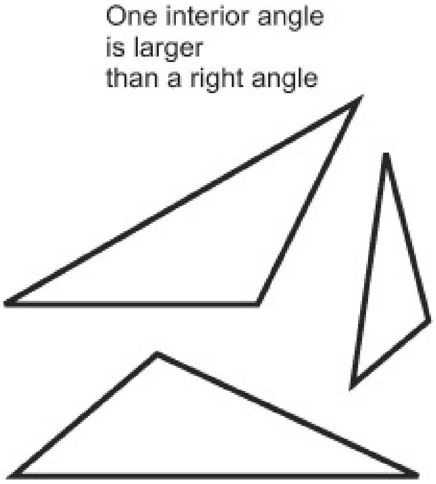

FIGURE 1-9. In an obtuse triangle, one angle measures more than a right angle.

Isosceles Triangle

Imagine a triangle with sides that have lengths *s*, *t*, and *u*. Let *x* represent the angle opposite the side of length *s*, let *y* represent the angle opposite the side of length *t*, and let *z* represent the angle opposi the side of length *u*. Now suppose that *at least one* of the following equations holds true:

$$s = t$$

$$t = u$$

$$s = u$$

$$x = y$$

$$y = z$$

$$x = z$$

Figure 1-10 shows an example of such a situation, where $s = t$. Whenever we find a triangle that has two sides of identical length, we call it an *isosceles triangle*. For sides and angles in the orientation of Fig. 1-10:

$$s = t \leftrightarrow x = y$$