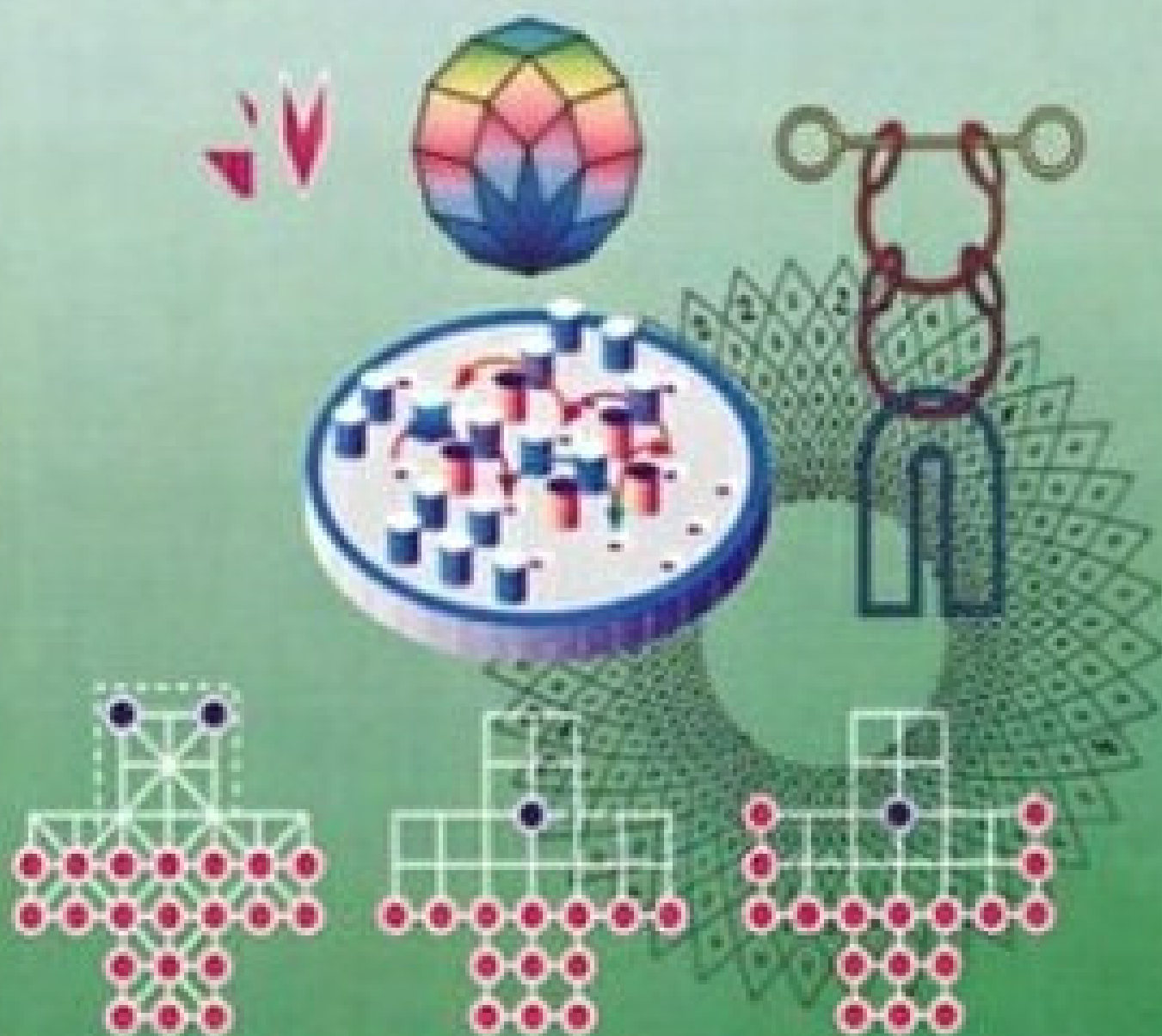


VOLUME 3

SECOND EDITION

# WINNING WAYS FOR YOUR MATHEMATICAL PLAYS



EDWYN R. BERLEKAMP • JOHN H. CONWAY • RICHARD K. GUY

---

## Winning Ways for Your Mathematical Plays, Volume 3

# Winning Ways

for Your Mathematical Plays



Volume 3, Second Edition

Elwyn R. Berlekamp, John H. Conway, Richard K. Guy



A K Peters  
Natick, Massachusetts

Editorial, Sales, and Customer Service Office

A K Peters, Ltd.  
63 South Avenue  
Natick, MA 01760

Copyright © 2003 by A K Peters, Ltd.

All rights reserved. No part of the material protected by this copyright notice may be reproduced or utilized in any form, electronic or mechanical, including photocopying, recording, or by any information storage and retrieval system, without written permission from the copyright owner.

**Library of Congress Cataloging-in-Publication Data**

Berlekamp, Elwyn R.

Winning Ways for your mathematical plays / Elwyn Berlekamp, John H. Conway,  
Richard Guy.--2nd ed.

p. cm.

Includes bibliographical references and index.

ISBN 1-56881-130-6 (v. 1) – ISBN 1-56881-142-X (v. 2) – ISBN 1-56881-143-8 (v. 3) –  
ISBN 1-56881-144-6 (v. 4) (alk.paper)

1. Mathematical recreations. I. Conway, John Horton. II. Guy Richard K. III. Title.

QA95 .B446 2000  
739.7'4--dc21

00-048541

Printed in Canada

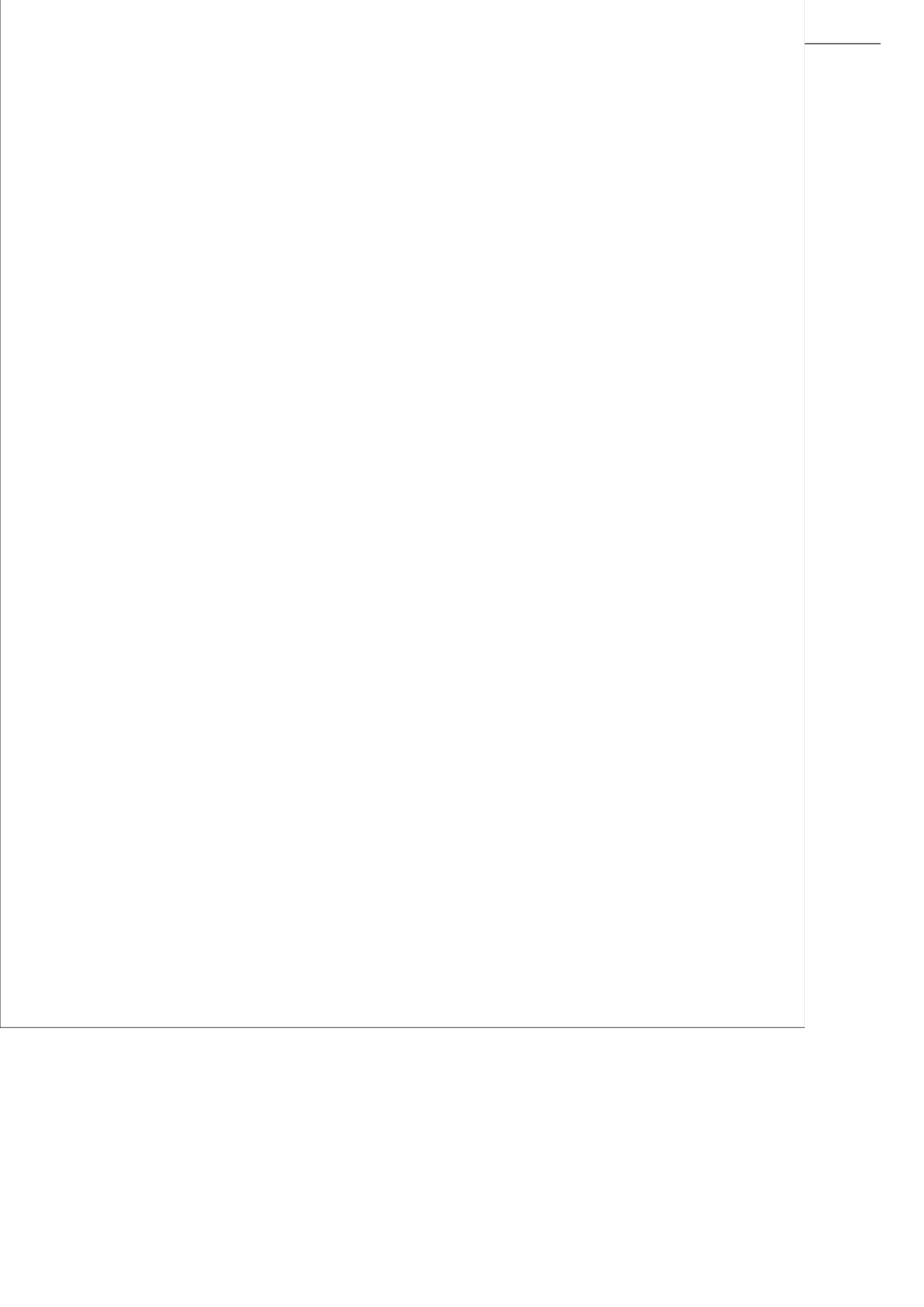
07 06 05 04 03

10 9 8 7 6 5 4 3 2 1

---

To Martin Gardner

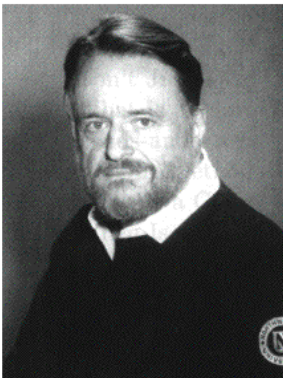
who has brought more mathematics to more millions than anyone else





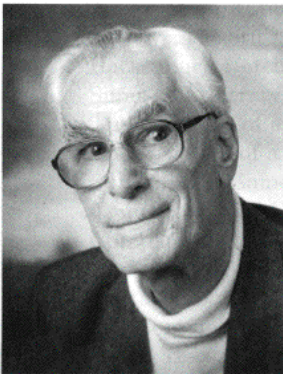
Elwyn Berlekamp was born in Dover, Ohio, on September 6, 1940. He has been Professor of Mathematics and of Electrical Engineering/Computer Science at UC Berkeley since 1971. He has also been active in several technology business ventures. In addition to writing many journal articles and several books, Berlekamp also has 12 patented inventions, mostly dealing with algorithms for synchronization and error correction.

He is a member of the National Academy of Sciences, the National Academy of Engineering, and the American Academy of Arts and Sciences. From 1994 to 1998, he was chairman of the board of trustees of the Mathematical Sciences Research Institute (MSRI).



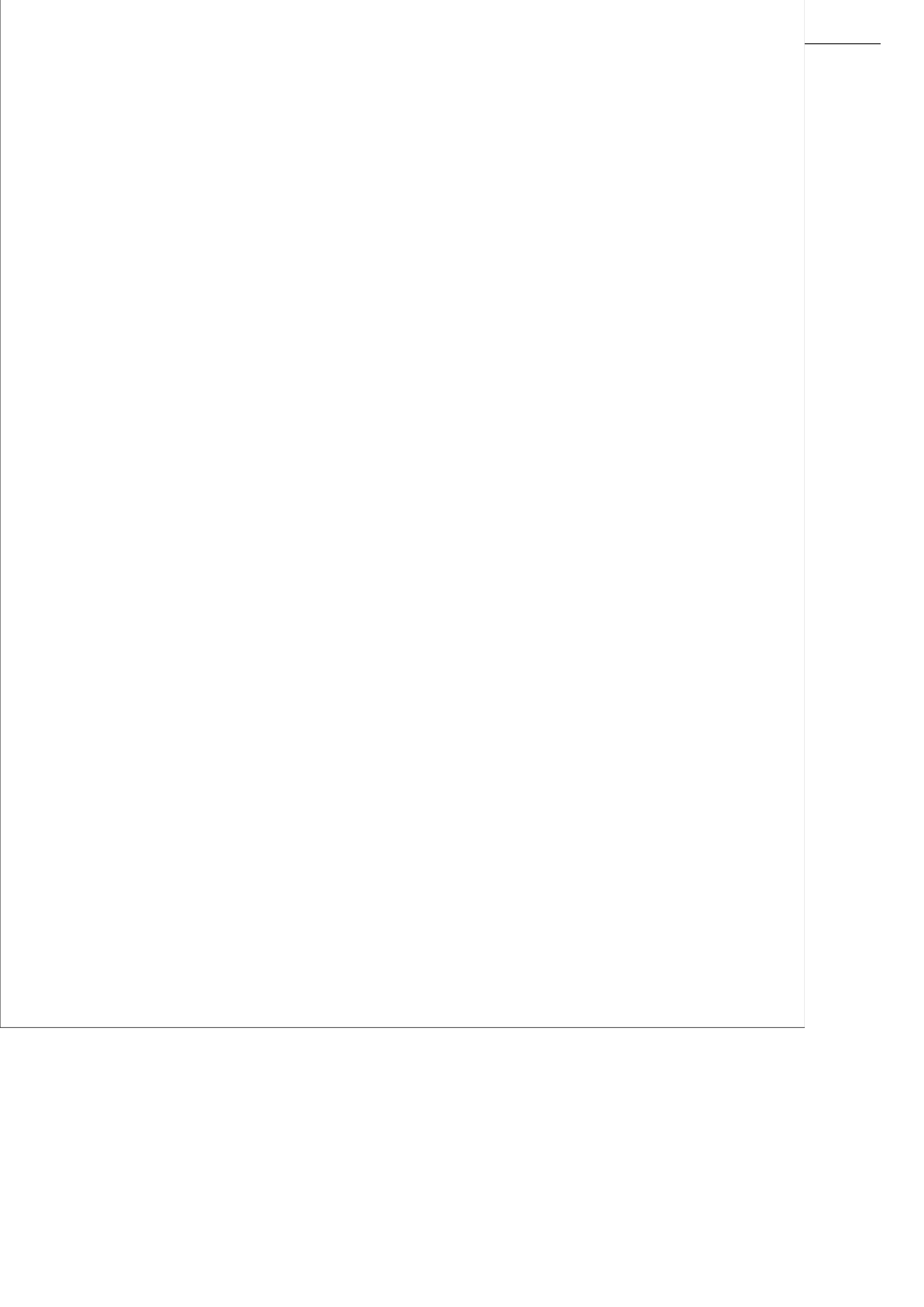
John H. Conway was born in Liverpool, England, on December 26, 1937. He is one of the preeminent theorists in the study of finite groups and the mathematical study of knots, and has written over 10 books and more than 140 journal articles.

Before joining Princeton University in 1986 as the John von Neumann Distinguished Professor of Mathematics, Conway served as professor of mathematics at Cambridge University, and remains an honorary fellow of Caius College. The recipient of many prizes in research and exposition, Conway is also widely known as the inventor of the Game of Life, a computer simulation of simple cellular "life," governed by remarkably simple rules.



Richard Guy was born in Nuneaton, England, on September 30, 1916. He has taught mathematics at many levels and in many places—England, Singapore, India, and Canada. Since 1965 he has been Professor of Mathematics at the University of Calgary, and is now Faculty Professor and Emeritus Professor. The university awarded him an Honorary Degree in 1991. He was Noyce Professor at Grinnell College in 2000.

He continues to climb mountains with his wife, Louise, and they have been patrons of the Association of Canadian Mountain Guides' Ball and recipients of the A. O. Wheeler award for Service to the Alpine Club of Canada.





# Contents

Preface to the Second Edition	xvii
Preface to the Original Edition	xviii
Games in Clubs!	xxi
14 Turn and Turn About	461
Turning Turtles . . . . .	461
Mock Turtles . . . . .	463
Odious and Evil Numbers . . . . .	463
Moebius, Mogul and Gold Moidores . . . . .	464
The Mock Turtle Theorem . . . . .	464
Why Moebius? . . . . .	466
Mogul . . . . .	467
Motley . . . . .	469
Twins, Triplets, Etc. . . . .	469
The Ruler Game . . . . .	469
Circumscribed Games . . . . .	470
Turnips (or Ternups) . . . . .	470
Grunt . . . . .	472
Sym . . . . .	473
Two-Dimensional Turning Games . . . . .	473
Acrostic Twins . . . . .	473
Turning Corners . . . . .	473
Nim-Multiplication . . . . .	475
Swirling Tartans . . . . .	476
The Tartan Theorem . . . . .	477
Rugs, Carpets, Windows and Doors . . . . .	478

Acrostic Games . . . . .	482
Stripping and Streaking . . . . .	483
Uglification and Derision . . . . .	483
Unlocking Doors . . . . .	488
Sparring, Boxing and Fencing . . . . .	488
“Coins” (or Heaps) with Infinitely Many (or $2^{2^N}$ ) “Sides” . . . . .	488
References and Further Reading . . . . .	489
<b>15 Chips and Strips . . . . .</b>	<b>491</b>
The Silver Dollar Game . . . . .	491
Profit from Gaming Tables . . . . .	492
Antonim . . . . .	493
Synonim . . . . .	494
Simonim . . . . .	496
Staircase Fives . . . . .	499
Twopins (pronounced “Tuppins”) . . . . .	500
Cram . . . . .	502
Welter’s Game . . . . .	506
Four-Coin Welter is Just Nim . . . . .	507
And So’s Three-Coin Welter! . . . . .	507
The Congruence Modulo 16 . . . . .	507
Frieze Patterns . . . . .	509
Inverting the Welter Function . . . . .	510
The Abacus Positions . . . . .	512
The Abacus Strategy . . . . .	513
The Misère Form of Welter’s Game . . . . .	514
Kotzig’s Nim . . . . .	515
Fibonacci Nim . . . . .	517
More Generally Bounded Nim . . . . .	517
Epstein’s Put-or-Take-a-Square Game . . . . .	518
Tribulations and Fibulations . . . . .	520
Third One Lucky . . . . .	520
Hickory, Dickory, Dock . . . . .	521
D.U.D.E.N.E.Y . . . . .	521
Strings of Pearls . . . . .	522
Schuhstrings . . . . .	523
The Princess and the Roses . . . . .	524
One-Step, Two-Step . . . . .	529
More on Subtraction Games . . . . .	529
Smallest Nim, Largest Nim . . . . .	532
Moore’s $\text{Nim}_k$ . . . . .	533
The More the Merrier . . . . .	533
Moore and More . . . . .	534
Not with a Bang but a Whim . . . . .	534



Did You Win the Silver Dollar? . . . . .	535
How Was Your Arithmetic? . . . . .	535
In Put-or-Take-a-Square , 92 Is an $\mathcal{N}$ -Position . . . . .	535
Tribulations and Fibulations . . . . .	535
Our Code of Behavior for Princes . . . . .	537
References and Further Reading . . . . .	538
16 Dots-and-Boxes . . . . .	541
Double-Dealing Leads to Double-Crosses . . . . .	543
How Long Is “Long”? . . . . .	546
The 4-Box Game . . . . .	547
The 9-Box Game . . . . .	549
The 16-Box Game . . . . .	549
Other Shapes of Board . . . . .	550
Dots-and-Boxes and Strings-and-Coins . . . . .	550
Nimstring . . . . .	552
Why Long Is Long . . . . .	554
To Take or Not to Take a Coin in Nimstring . . . . .	555
Sprague-Grundy Theory for Nimstring Graphs . . . . .	556
All Long Chains Are the Same . . . . .	561
Which Mutations Are Harmless? . . . . .	562
Chopping and Changing . . . . .	564
Vines . . . . .	564
Dots + Doublecrosses = Turns . . . . .	571
How Dodie Can Win the 4-Box Game . . . . .	572
When Is it Best to Lose Control? . . . . .	574
Computing the Values of Vines . . . . .	575
Loony Endgames Are NP-Hard . . . . .	577
Solutions to Dots-and-Boxes Problems . . . . .	578
Some More Nimstring Values . . . . .	581
Nimbers for Nimstring Arrays . . . . .	581
References and Further Reading . . . . .	584
17 Spots and Sprouts . . . . .	585
Rims . . . . .	585
Rails . . . . .	586
Loops-and-Branched . . . . .	586
Contours . . . . .	587
Lucasta . . . . .	588
A Child’s Guide to Normal Lucasta . . . . .	589
The Misère Form of Lucasta . . . . .	590
The Positions (7, 3, 1) and (11, 1, 1) . . . . .	594
Cabbages; or Bugs, Caterpillars and Cocoons . . . . .	597
Jocasta . . . . .	597

Sprouts . . . . .	598
Brussels Sprouts . . . . .	603
Stars-and-Stripes . . . . .	603
Bushenhack . . . . .	604
Genetic Codes for Nim . . . . .	605
Bushenhack Positions Have Genetic Codes! . . . . .	606
Von Neumann Hackenbush . . . . .	606
The Joke in Jocasta . . . . .	607
The Worm in Brussels Sprouts . . . . .	607
Bushenhack . . . . .	607
References and Further Reading . . . . .	607
<b>18 The Emperor and His Money</b> . . . . .	<b>609</b>
Sylver Coinage . . . . .	610
How Long Will It Last? . . . . .	610
Some Openings Are Bad . . . . .	611
Are All Openings Bad? . . . . .	613
Not All Openings Are Bad . . . . .	615
Strategy Stealing . . . . .	616
Quiet Ends . . . . .	617
Doubling and Tripling? . . . . .	620
Halving and Thirding? . . . . .	620
Finding the Right Combinations . . . . .	621
What Shall I do When $g$ is Two . . . . .	626
The Great Unknown . . . . .	629
Are Outcomes Computable? . . . . .	630
The Etiquette of Sylver Coinage . . . . .	631
Chomp . . . . .	632
Zig-Zag . . . . .	632
More Cliques for Sylver Coinage . . . . .	635
5-Pairs . . . . .	635
Positions Containing 6 . . . . .	636
Sylver Coinage Has Infinite Nim-Values . . . . .	636
A Few Final Questions . . . . .	640
References and Further Reading . . . . .	640
<b>19 The King and the Consumer</b> . . . . .	<b>641</b>
Chessgo, Kinggo and Dukego . . . . .	641
Quadruphage . . . . .	642
The Angel and the Square-Eater . . . . .	643
Strategy and Tactics . . . . .	643
Dukego . . . . .	644
The Edge Attack . . . . .	646
The Edge Defence . . . . .	647

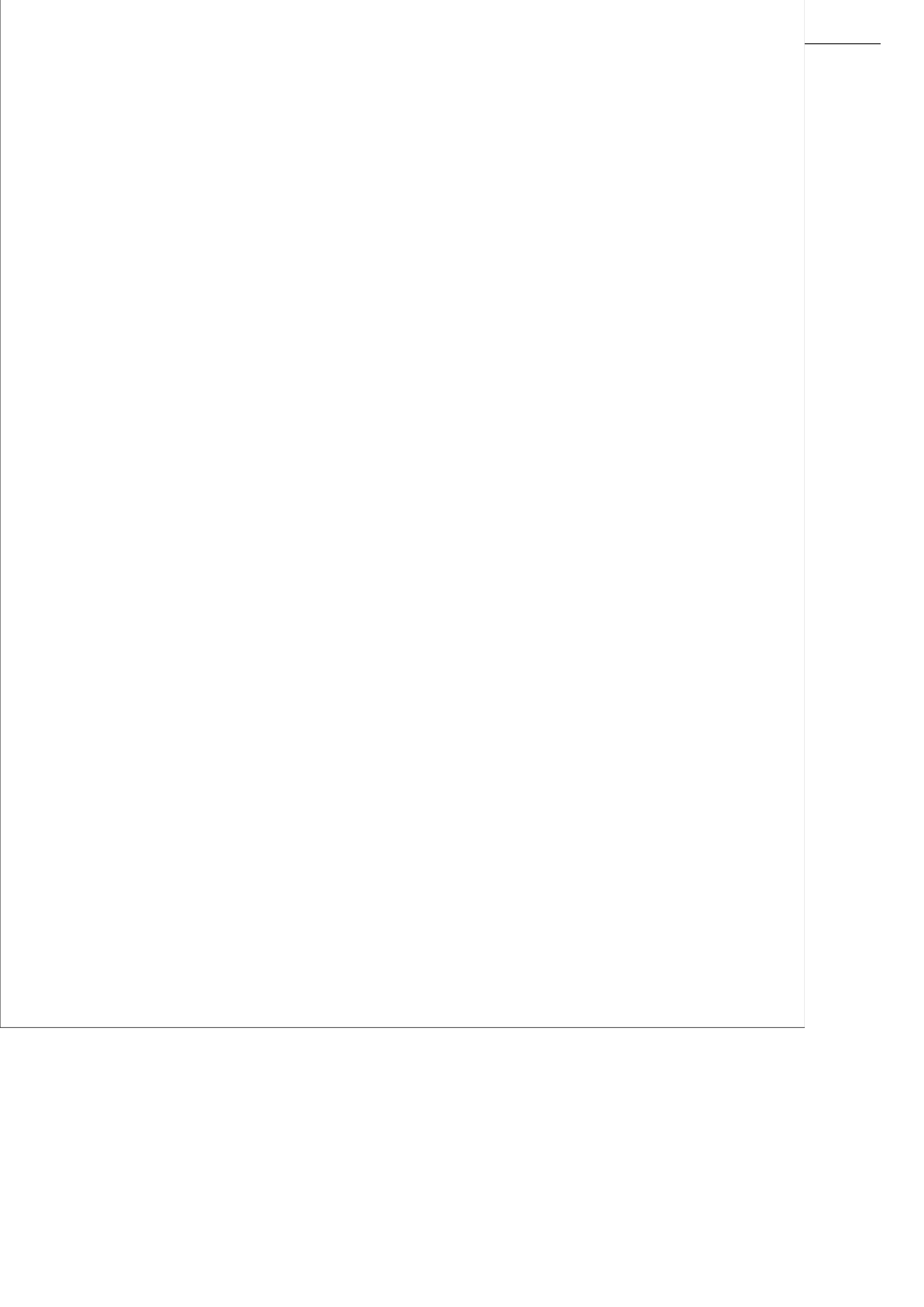


A Memoryless Edge Defence . . . . .	648
The Edge-Corner Attack . . . . .	650
Strategic and Tactical Stones . . . . .	652
Corner Tactics . . . . .	654
Defence on Large Square Boards . . . . .	657
The $33 \times 33$ Board . . . . .	658
The Centred King . . . . .	658
Leaving the Central Region . . . . .	658
The Cornered King . . . . .	661
The Sidelined King . . . . .	661
How Chas. Can Win on a $34 \times 34$ Board . . . . .	663
Rectangular Boards . . . . .	664
Many-Dimensional Angels . . . . .	665
Games of Encirclement . . . . .	665
Wolves-and-Sheep . . . . .	665
Tablut . . . . .	666
Saxon Hnefatafl . . . . .	666
King and Rook Versus King . . . . .	667
References and Further Reading . . . . .	668
20 Fox and Geese . . . . .	669
Some Properties of Our Strategy . . . . .	672
What Is the Value of Fox-and-Geese? . . . . .	673
Fox-Flocks-Fox . . . . .	673
Towards Greater Precision . . . . .	673
The Indefinite Board . . . . .	674
How the Geese Survive Triumphantly . . . . .	675
Eight Exciting Escapades . . . . .	677
Tactics and Startegy for the Fox . . . . .	681
Fox's Play on Definite Boards . . . . .	686
The Scrimmage Sequence . . . . .	687
Values of the Initial Positions . . . . .	688
Sophisticated New Software . . . . .	690
Values of FOXTAC Positions . . . . .	692
High Region . . . . .	692
Welton's Region of Value <b>2over</b> . . . . .	693
Lower "Delta" Region . . . . .	697
Initial Values . . . . .	697
Parity in the High Scrimmage Region . . . . .	697
Early Values . . . . .	700
The <i>abc</i> 's to Trifurcate the Two-Ish Transition . . . . .	701
Extending the Backbone to Lower Altitudes . . . . .	702
Early Values of Other Formations . . . . .	702
Resolving the Exceptions . . . . .	704

Museums . . . . .	705
Solutions to Problems . . . . .	709
Open Problems . . . . .	710
Maharajah and Sepoys . . . . .	710
References and Further Reading . . . . .	710
<b>21 Hare and Hounds</b> . . . . .	<b>711</b>
The French Military Hunt . . . . .	711
Two Trial Games . . . . .	713
History . . . . .	713
The Different Kinds of Place . . . . .	713
The Opposition . . . . .	714
When Has the Hare Escaped? . . . . .	715
Losing the Opposition . . . . .	716
A Strategy for the Hare . . . . .	718
On the Small Board . . . . .	721
On the Medium and Larger Boards . . . . .	722
Answers to Questions . . . . .	725
A Sound Bound for a Hound? . . . . .	725
All Is Found for the Small Board Hound . . . . .	725
Proof of the Thirty-One Theorem . . . . .	729
References and Further Reading . . . . .	729
<b>22 Lines and Squares</b> . . . . .	<b>731</b>
Tit-Tat-Toe, My First Go, Three Jolly Butcher Boys All in a Row . . . . .	731
Magic Fifteen . . . . .	732
Spit Not So, Fat Fop, as if in Pan! . . . . .	732
Jam . . . . .	732
How Long Can You Fool Your Friends? . . . . .	733
Analysis of Tic-Tac-Toe . . . . .	733
Ovid's Game, Hopscotch, Les Pendus . . . . .	736
Six Men's Morris . . . . .	737
Nine Men's Morris . . . . .	737
Three Up . . . . .	737
Four-in-a-Row . . . . .	737
Five-in-a-Row . . . . .	738
Go-Moku . . . . .	740
Six, Seven, Eight, Nine, . . . , in a Row . . . . .	740
$n$ -Dimensional $k$ -in-a-Row . . . . .	742
Strategy Stealing in Tic-Tac-Toe Games . . . . .	743
Hex . . . . .	744
Bridgit . . . . .	744
How Does the First Player Win? . . . . .	744
The Shannon Switching Game . . . . .	744



The Black Path Game . . . . .	746
Lewthwaite's Game . . . . .	747
Meander . . . . .	747
Winners and Losers . . . . .	748
Dodgem . . . . .	749
Dodgerydoo . . . . .	751
Philosopher's Football . . . . .	752
Count foxy words And stay awake Using lively wit . . . . .	756
Amazons . . . . .	756
Checkers . . . . .	757
Chess . . . . .	757
Cherries . . . . .	757
Clobber . . . . .	757
Go . . . . .	758
Konane . . . . .	759
Reversi-Othello . . . . .	760
Scrabble . . . . .	760
Shogi . . . . .	760
Sowing Games . . . . .	761
Answers to Problems . . . . .	761
References and Further Reading . . . . .	762
Index to Volumes 1–3 . . . . .	770





# Preface to the Second Edition

In the first edition of *Winning Ways*, which appeared in 1982, we were able to make a rather sharp distinction between those games in Part I, to which the major theory of addition applied directly, and those games in Part 3, which seemed to require more specialized techniques. However, subsequent research by an increasingly large community of combinatorial game theorists has begun to blur this distinction. We now have many more games whose strategies depend both on the general theory of Volume 1 as well as on more specialized results. Introductions to many of these games and some illustrative problems have been added to this new edition. Those that did not readily fit elsewhere can be found in the new Extras to Chapter 22 at the end of this volume. This volume also includes a major revision of the original Chapter 20 on the game of Fox and Geese. Its enhanced variation, Fox-Flocks-Fox, provides compelling illustrations of some of the challenging problems that can now be solved by appropriately combining theories from Volumes 1, 2, and 3 with innovative computing algorithms.

This new edition owes much to the supportive efforts of numerous friends and colleagues, including Noam Elkies, Tom Ferguson, Aviezri Fraenkel, Martin Gardner, Sol Golomb, Al Hales, Greg Kuperberg, Silvio Levy, Donald Knuth, Martin Kutz, Greg Martin, Victor Meally, Richard Nowakowski, Hilarie Orman, Marc Paulhus, Ed Pegg, Michael Reid, Thea van Roode, Katherine Scott, George Sicherman, Aaron Siegel, Neil Sloane, Sally Smith, William Spight, John Tromp, Jonathan Welton, Julian West, David Wilson, and David Wolfe, and to the very professional yet kindly support of our publishers, Alice and Klaus Peters.

Elwyn Berlekamp, University of California, Berkeley  
John Conway, Princeton University  
Richard Guy, The University of Calgary, Canada

June 23, 2003

# Preface to the Original Edition

Does a book need a Preface? What more, after fifteen years of toil, do three talented authors have to add.

We can reassure the bookstore browser, “Yes, this is just the book you want!”

We can direct you, if you want to know quickly what’s in the book, to page xx. This in turn directs you to volumes 1,2,3 and 4.

We can supply the reviewer, faced with the task of ploughing through nearly a thousand information-packed pages, with some pithy criticisms by indicating the horns of the polylemma the book finds itself on. It is not an encyclopedia. It is encyclopedic, but there are still too many games missing for it to claim to be complete. It is not a book on recreational mathematics because there’s too much serious mathematics in it. On the other hand, for us, as for our predecessors Rouse Ball, Dudeney, Martin Gardner, Kraitchik, Sam Loyd, Lucas, Tom O’Beirne and Fred. Schuh, mathematics itself is a recreation. It is not an undergraduate text, since the exercises are not set out in an orderly fashion, with the easy ones at the beginning. They are there though, and with the hundred and sixty-three mistakes we’ve left in, provide plenty of opportunity for reader participation. So don’t just stand back and admire it, work of art though it is. It is not a graduate text, since it’s too expensive and contains far more than any graduate student can be expected to learn. But it does carry you to the frontiers of research in combinatorial game theory and the many unsolved problems will stimulate further discoveries.

We thank Patrick Browne for our title. This exercised us for quite a time. One morning, while walking to the university, John and Richard came up with “Whose game?” but realized they couldn’t spell it (there are three tooze in English) so it became a one-line joke on line one of the text. There isn’t room to explain all the jokes, not even the fifty-nine private ones (each of our birthdays appears more than once in the book).

Omar started as a joke, but soon materialized as Kimberly King. Louise Guy also helped with proof-reading, but her greater contribution was the hospitality which enabled the three of us to work together on several occasions. Louise also did technical typing after many drafts had been made by Karen McDermid and Betty Teare.

Our thanks for many contributions to content may be measured by the number of names in the index. To do real justice would take too much space. Here’s an abridged list of helpers: Richard Austin, Clive Bach, John Beasley, Aviezri Fraenkel, David Fremlin, Solomon Golomb, Steve Grantham, Mike Guy, Dean Hickerson, Hendrick Lenstra, Richard Nowakowski, Anne Scott, David Seal, John Selfridge, Cedric Smith and Steve Tschantz.

No small part of the reason for the assured success of the book is owed to the well-informed and sympathetic guidance of Len Cegielka and the willingness of the staff of Academic Press and of Page Bros. to adapt to the idiosyncrasies of the authors, who grasped every opportunity to modify grammar, strain semantics, pervert punctuation, alter orthography, tamper with traditional typography and commit outrageous puns and inside jokes.

Thanks also to the the Isaak Walton Killam Foundation for Richard's Resident Fellowship at The University of Calgary during the compilation of a critical draft, and to the National (Science & Engineering) Research Council of Canada for a grant which enabled Elwyn and John to visit him more frequently than our widely scattered habitats would normally allow.

And thank you, Simon!

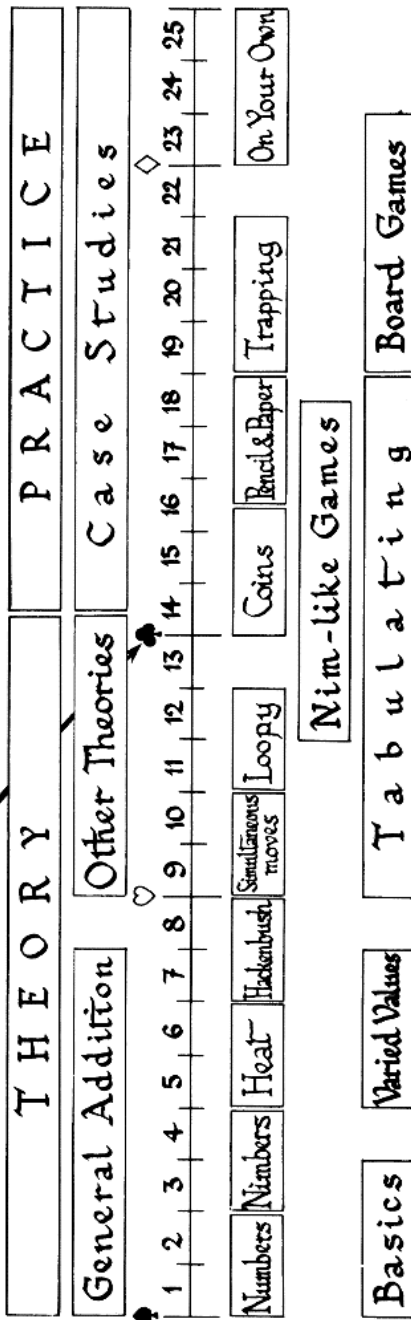
*University of California, Berkeley, CA 94720 Elwyn Berlekamp*  
*University of Cambridge, England, CB2 1SB John Conway*  
*University of Calgary, Canada, T2N 1N4 Richard Guy*

You are  
now here

If you want to know roughly what's elsewhere,  
turn to the little notes about our four main themes:

- Adding Games ... ♣ ... page 1
- Bending the Rules ... ♡ ... page 277
- Case Studies ... ♣ ... page 461
- Doing It Yourself ... ♠ ... page 803

There are a number of other connexions between various chapters of the book:



However, you should be able to pick any chapter and read almost all of it  
without reference to anything earlier, except perhaps the basic ideas at the start of the book.



## Games in Clubs!

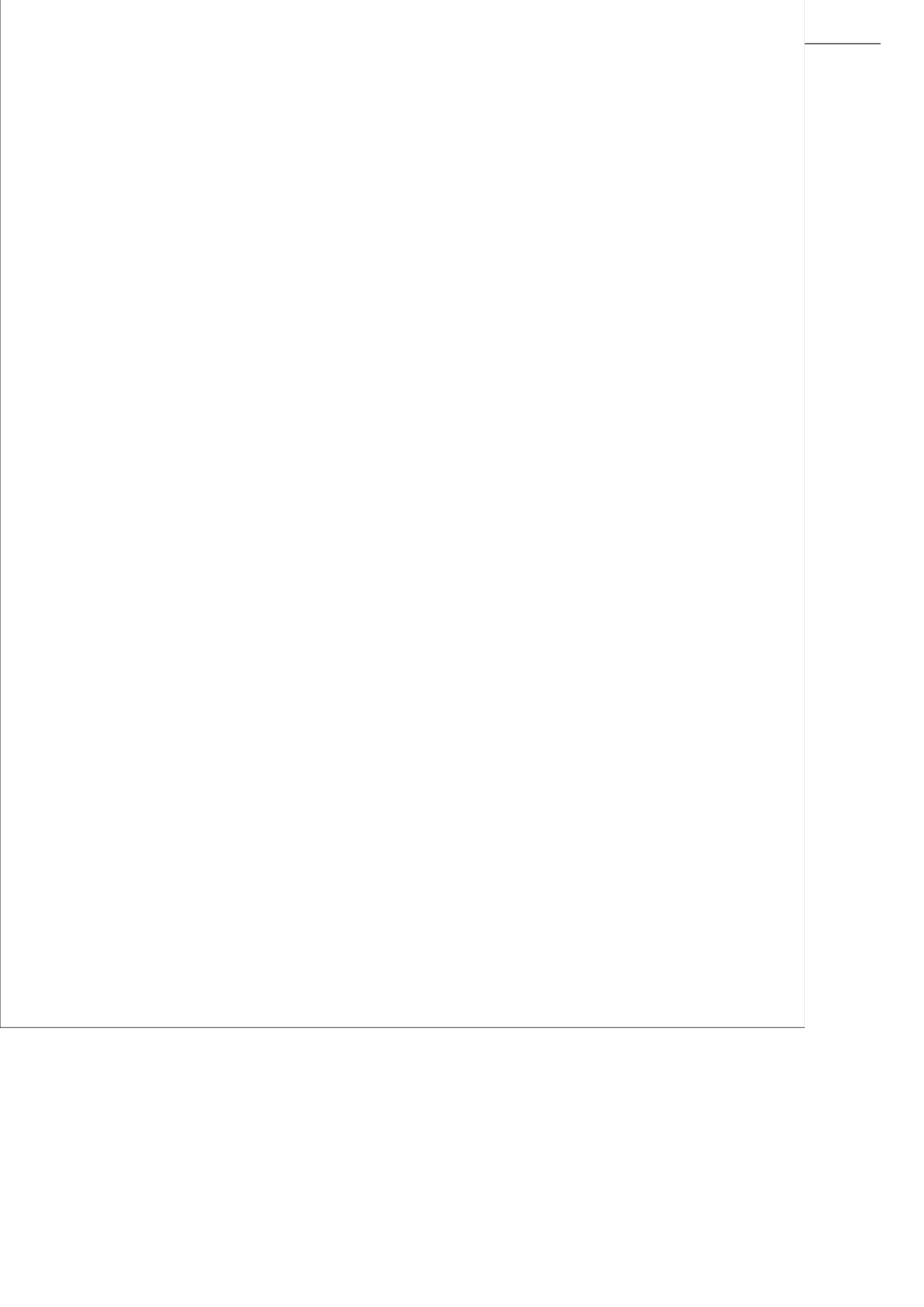
To be an Englishman is to belong  
to the most exclusive club there is.  
Ogden Nash, *England Expects*.

There are lots of games for which the theories we've now developed are useful, and even more for which they're not, and we've grouped them into clubs according to how you play them.

First some games you can play with coins, either by turning them over (Chapter 14) or moving them along strips or about in heaps (Chapter 15).

Then games for which you'll need pencil and paper, perhaps to draw straight lines (Chapter 16), or curved ones (Chapter 17) or merely to do the calculations in Chapter 18.

And for board games we have three case studies in which one player wins by trapping his opponent (Chapters 19, 20, 21) and finally many more which are usually won by the first player to establish some kind of winning configuration (Chapter 22).



## Turn and Turn About

Because I do not hope to turn again  
Because I do not hope  
Because I do not hope to turn.

T. S. Eliot, *Ash Wednesday*, I.

Open not thine heart to every man, lest he requite thee  
with a shrewd turn.

Ecclesiasticus, 8:19.

These games, based on an idea of H. W. Lenstra, are similar in that they all involve turning things over, but we shall see that they call for a variety of strategies.

### Turning Turtles

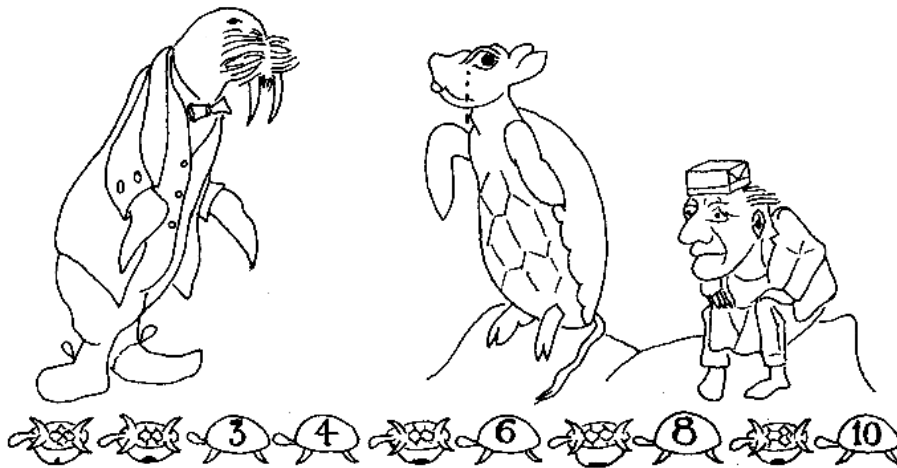


Figure 1. Playing Turning Turtles.

In Fig. 1 the Walrus and the Carpenter are playing a rather cruel game. At each move a player must put one turtle on its back and may also turn over any single turtle to the left of it. This second turtle, unlike the first, may be turned either onto its feet or onto its back. The player wins who turns the last turtle upside-down. Which turtles should the Walrus (*l.*) turn?

Like most readers of this book, he wearily suspects another disguise for Nim. Here only turtles 3, 4, 6, 8 and 10 are on their feet, and since the nim-sum of 3, 4 and 6 is 1, he may turn 10 onto its back and 9 onto its feet, producing 3, 4, 6, 8, 9, a  $\mathcal{P}$ -position since  $8 \oplus 9 = 1$ . The Carpenter (*r.*) responds by turning 8 and 5 producing the position 3, 4, 5, 6, 9 as in Fig. 2.



Figure 2. After the Carpenter's Reply.

In Nim there is only one good move from this position—reduce 9 to 4, so as to produce, 3, 4, 4, 5, 6, which, since two equal Nim heaps may be cancelled, is much the same as 3, 5, 6, which the Walrus reaches by turning both 9 and 4 on their backs (Fig. 3).

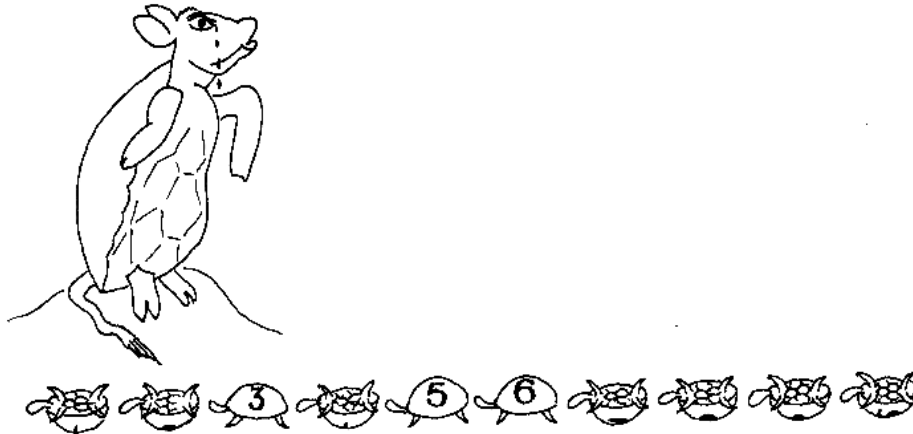


Figure 3. How the Walrus Won.

Nim moves become turtle turns as follows. We reduce a heap to a size not already present by turning one turtle on its back and putting another on its feet, as in the Walrus's opening move. If a heap of the reduced size is already present, we turn two turtles on their backs as in the Walrus's response to the Carpenter's move (cancelling two equal heaps). To eliminate a heap entirely, we merely turn the appropriate turtle. So since 4, 6, 8, 10 is a  $\mathcal{P}$ -position, the Walrus could have won from Fig. 1 by just turning turtle 3.

Since all our turning games are impartial, they are solved by computing the nim-values, and often may be thought of as heap games in disguise; but many games with interesting theories are more naturally suggested by the turning version.



- [Wages So Low You'll Freak pdf, azw \(kindle\)](#)
- [How the Fox Got His Color - CÃ³mo el zorro obtuvo su color \(English-Spanish Bilingual Edition\) pdf, azw \(kindle\), epub](#)
- [read Exploration of the Solar System by Infrared Remote Sensing \(2nd Edition\)](#)
- [download online Dante in Love pdf, azw \(kindle\), epub, doc, mobi](#)
- [read online A Highlander For Christmas \(Draycott Abbey, Book 7\)](#)
- [click American Colossus: The Triumph of Capitalism, 1865-1900 for free](#)
  
- <http://creativebeard.ru/freebooks/Now-We-Are-Six.pdf>
- <http://omarnajmi.com/library/DB2-10-5-with-BLU-Acceleration--New-Dynamic-In-Memory-Analytics-for-the-Era-of-Big-Data.pdf>
- <http://unpluggedtv.com/lib/Exploration-of-the-Solar-System-by-Infrared-Remote-Sensing--2nd-Edition-.pdf>
- <http://deltaphenomics.nl/?library/Mahatma-Gandhi--Critical-Lives-.pdf>
- <http://www.freightunlocked.co.uk/lib/A-Highlander-For-Christmas--Draycott-Abbey--Book-7-.pdf>
- <http://reseauplatoparis.com/library/Winter-Kills.pdf>