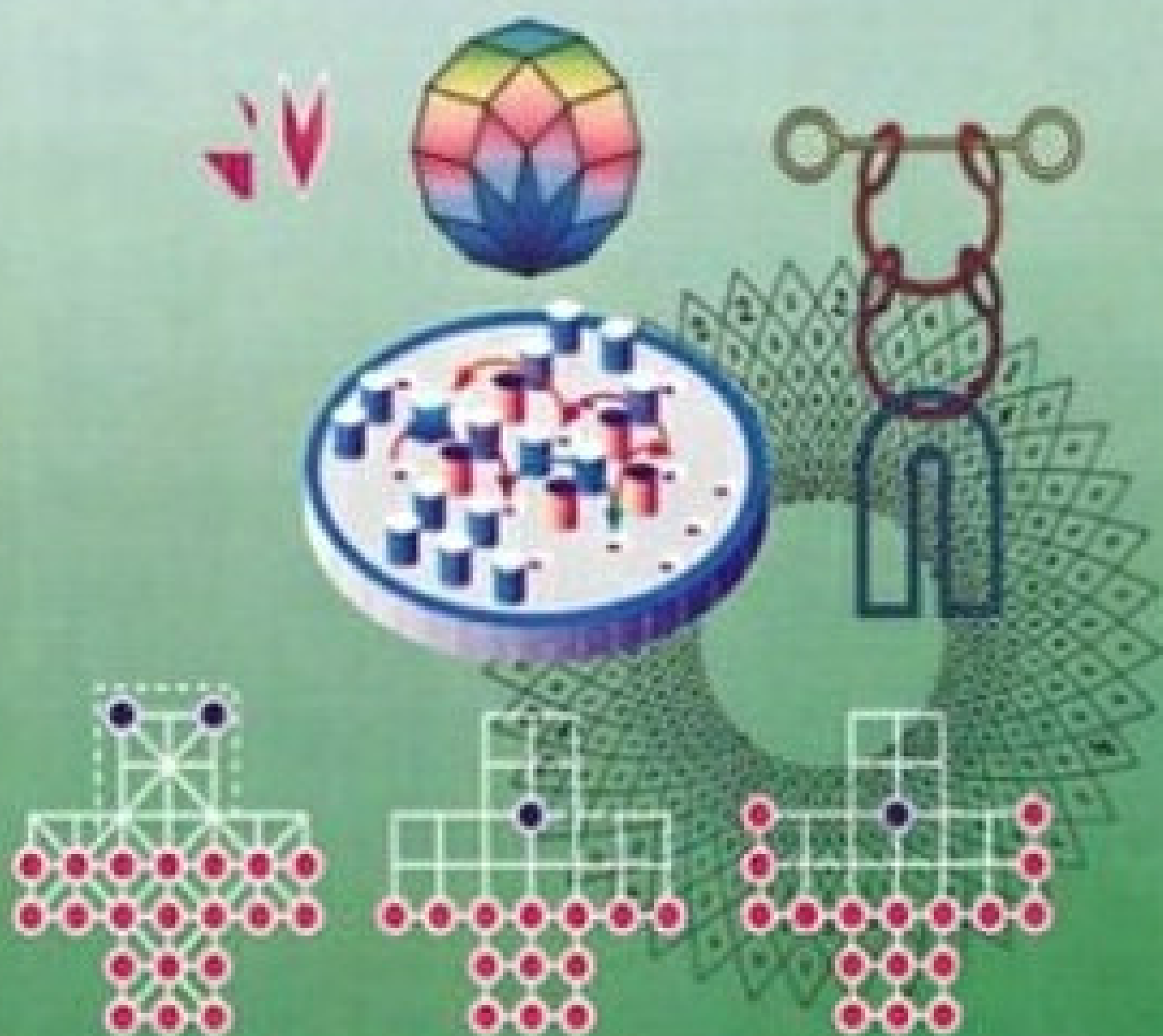


VOLUME 3

SECOND EDITION

WINNING WAYS FOR YOUR MATHEMATICAL PLAYS



EDWYN R. BERLEKAMP • JOHN H. CONWAY • RICHARD K. GUY

Winning Ways for Your Mathematical Plays, Volume 3

Winning Ways

for Your Mathematical Plays



Volume 3, Second Edition

Elwyn R. Berlekamp, John H. Conway, Richard K. Guy



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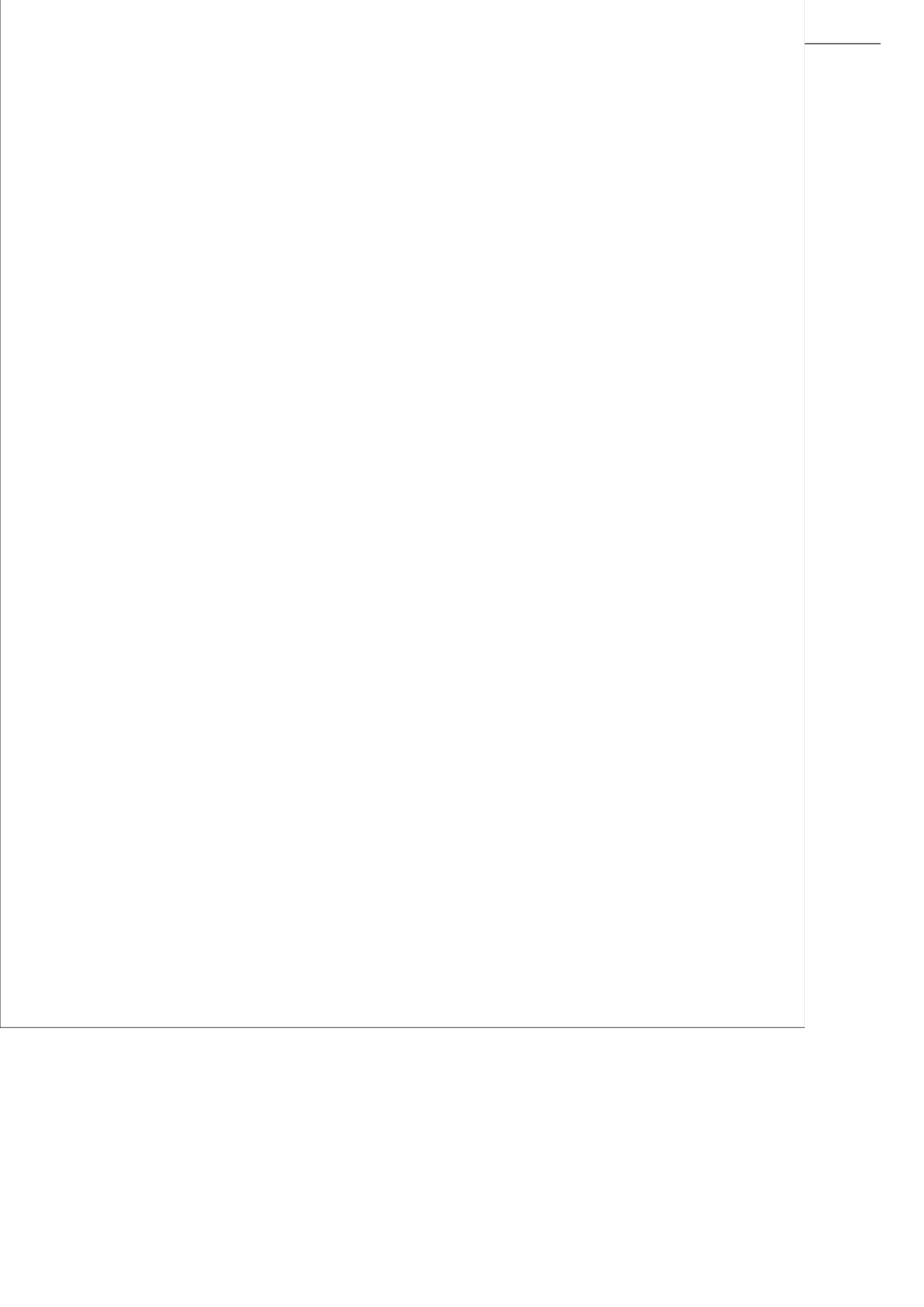
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To Martin Gardner

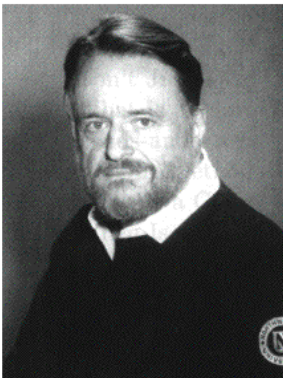
who has brought more mathematics to more millions than anyone else





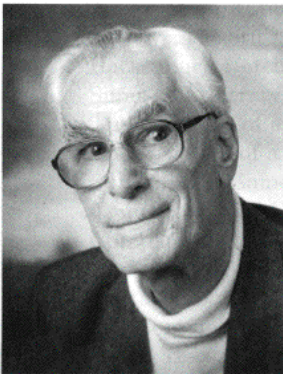
Elwyn Berlekamp was born in Dover, Ohio, on September 6, 1940. He has been Professor of Mathematics and of Electrical Engineering/Computer Science at UC Berkeley since 1971. He has also been active in several technology business ventures. In addition to writing many journal articles and several books, Berlekamp also has 12 patented inventions, mostly dealing with algorithms for synchronization and error correction.

He is a member of the National Academy of Sciences, the National Academy of Engineering, and the American Academy of Arts and Sciences. From 1994 to 1998, he was chairman of the board of trustees of the Mathematical Sciences Research Institute (MSRI).



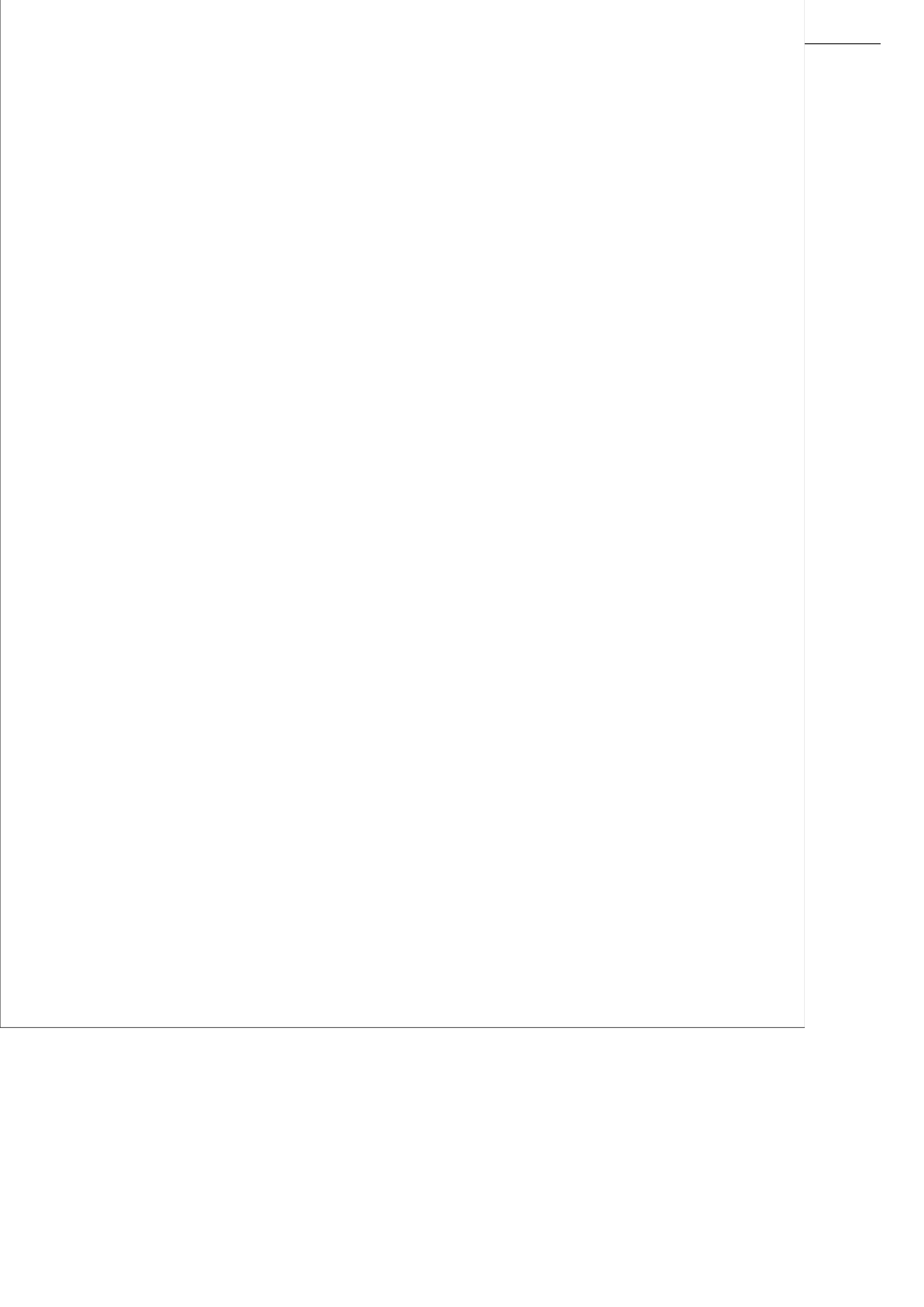
John H. Conway was born in Liverpool, England, on December 26, 1937. He is one of the preeminent theorists in the study of finite groups and the mathematical study of knots, and has written over 10 books and more than 140 journal articles.

Before joining Princeton University in 1986 as the John von Neumann Distinguished Professor of Mathematics, Conway served as professor of mathematics at Cambridge University, and remains an honorary fellow of Caius College. The recipient of many prizes in research and exposition, Conway is also widely known as the inventor of the Game of Life, a computer simulation of simple cellular "life," governed by remarkably simple rules.



Richard Guy was born in Nuneaton, England, on September 30, 1916. He has taught mathematics at many levels and in many places—England, Singapore, India, and Canada. Since 1965 he has been Professor of Mathematics at the University of Calgary, and is now Faculty Professor and Emeritus Professor. The university awarded him an Honorary Degree in 1991. He was Noyce Professor at Grinnell College in 2000.

He continues to climb mountains with his wife, Louise, and they have been patrons of the Association of Canadian Mountain Guides' Ball and recipients of the A. O. Wheeler award for Service to the Alpine Club of Canada.



Contents

Preface to the Second Edition	xvii
Preface to the Original Edition	xviii
Games in Clubs!	xxi
14 Turn and Turn About	461
Turning Turtles	461
Mock Turtles	463
Odious and Evil Numbers	463
Moebius, Mogul and Gold Moidores	464
The Mock Turtle Theorem	464
Why Moebius?	466
Mogul	467
Motley	469
Twins, Triplets, Etc.	469
The Ruler Game	469
Circumscribed Games	470
Turnips (or Ternups)	470
Grunt	472
Sym	473
Two-Dimensional Turning Games	473
Acrostic Twins	473
Turning Corners	473
Nim-Multiplication	475
Swirling Tartans	476
The Tartan Theorem	477
Rugs, Carpets, Windows and Doors	478

Acrostic Games	482
Stripping and Streaking	483
Uglification and Derision	483
Unlocking Doors	488
Sparring, Boxing and Fencing	488
“Coins” (or Heaps) with Infinitely Many (or 2^{2^N}) “Sides”	488
References and Further Reading	489
15 Chips and Strips	491
The Silver Dollar Game	491
Profit from Gaming Tables	492
Antonim	493
Synonim	494
Simonim	496
Staircase Fives	499
Twopins (pronounced “Tuppins”)	500
Cram	502
Welter’s Game	506
Four-Coin Welter is Just Nim	507
And So’s Three-Coin Welter!	507
The Congruence Modulo 16	507
Frieze Patterns	509
Inverting the Welter Function	510
The Abacus Positions	512
The Abacus Strategy	513
The Misère Form of Welter’s Game	514
Kotzig’s Nim	515
Fibonacci Nim	517
More Generally Bounded Nim	517
Epstein’s Put-or-Take-a-Square Game	518
Tribulations and Fibulations	520
Third One Lucky	520
Hickory, Dickory, Dock	521
D.U.D.E.N.E.Y	521
Strings of Pearls	522
Schuhstrings	523
The Princess and the Roses	524
One-Step, Two-Step	529
More on Subtraction Games	529
Smallest Nim, Largest Nim	532
Moore’s Nim_k	533
The More the Merrier	533
Moore and More	534
Not with a Bang but a Whim	534



Did You Win the Silver Dollar?	535
How Was Your Arithmetic?	535
In Put-or-Take-a-Square , 92 Is an \mathcal{N} -Position	535
Tribulations and Fibulations	535
Our Code of Behavior for Princes	537
References and Further Reading	538
16 Dots-and-Boxes	541
Double-Dealing Leads to Double-Crosses	543
How Long Is “Long”?	546
The 4-Box Game	547
The 9-Box Game	549
The 16-Box Game	549
Other Shapes of Board	550
Dots-and-Boxes and Strings-and-Coins	550
Nimstring	552
Why Long Is Long	554
To Take or Not to Take a Coin in Nimstring	555
Sprague-Grundy Theory for Nimstring Graphs	556
All Long Chains Are the Same	561
Which Mutations Are Harmless?	562
Chopping and Changing	564
Vines	564
Dots + Doublecrosses = Turns	571
How Dodie Can Win the 4-Box Game	572
When Is it Best to Lose Control?	574
Computing the Values of Vines	575
Loony Endgames Are NP-Hard	577
Solutions to Dots-and-Boxes Problems	578
Some More Nimstring Values	581
Nimbers for Nimstring Arrays	581
References and Further Reading	584
17 Spots and Sprouts	585
Rims	585
Rails	586
Loops-and-Branched	586
Contours	587
Lucasta	588
A Child’s Guide to Normal Lucasta	589
The Misère Form of Lucasta	590
The Positions (7, 3, 1) and (11, 1, 1)	594
Cabbages; or Bugs, Caterpillars and Cocoons	597
Jocasta	597

Sprouts	598
Brussels Sprouts	603
Stars-and-Stripes	603
Bushenhack	604
Genetic Codes for Nim	605
Bushenhack Positions Have Genetic Codes!	606
Von Neumann Hackenbush	606
The Joke in Jocasta	607
The Worm in Brussels Sprouts	607
Bushenhack	607
References and Further Reading	607
18 The Emperor and His Money	609
Sylver Coinage	610
How Long Will It Last?	610
Some Openings Are Bad	611
Are All Openings Bad?	613
Not All Openings Are Bad	615
Strategy Stealing	616
Quiet Ends	617
Doubling and Tripling?	620
Halving and Thirding?	620
Finding the Right Combinations	621
What Shall I do When g is Two	626
The Great Unknown	629
Are Outcomes Computable?	630
The Etiquette of Sylver Coinage	631
Chomp	632
Zig-Zag	632
More Cliques for Sylver Coinage	635
5-Pairs	635
Positions Containing 6	636
Sylver Coinage Has Infinite Nim-Values	636
A Few Final Questions	640
References and Further Reading	640
19 The King and the Consumer	641
Chessgo, Kinggo and Dukego	641
Quadrphage	642
The Angel and the Square-Eater	643
Strategy and Tactics	643
Dukego	644
The Edge Attack	646
The Edge Defence	647

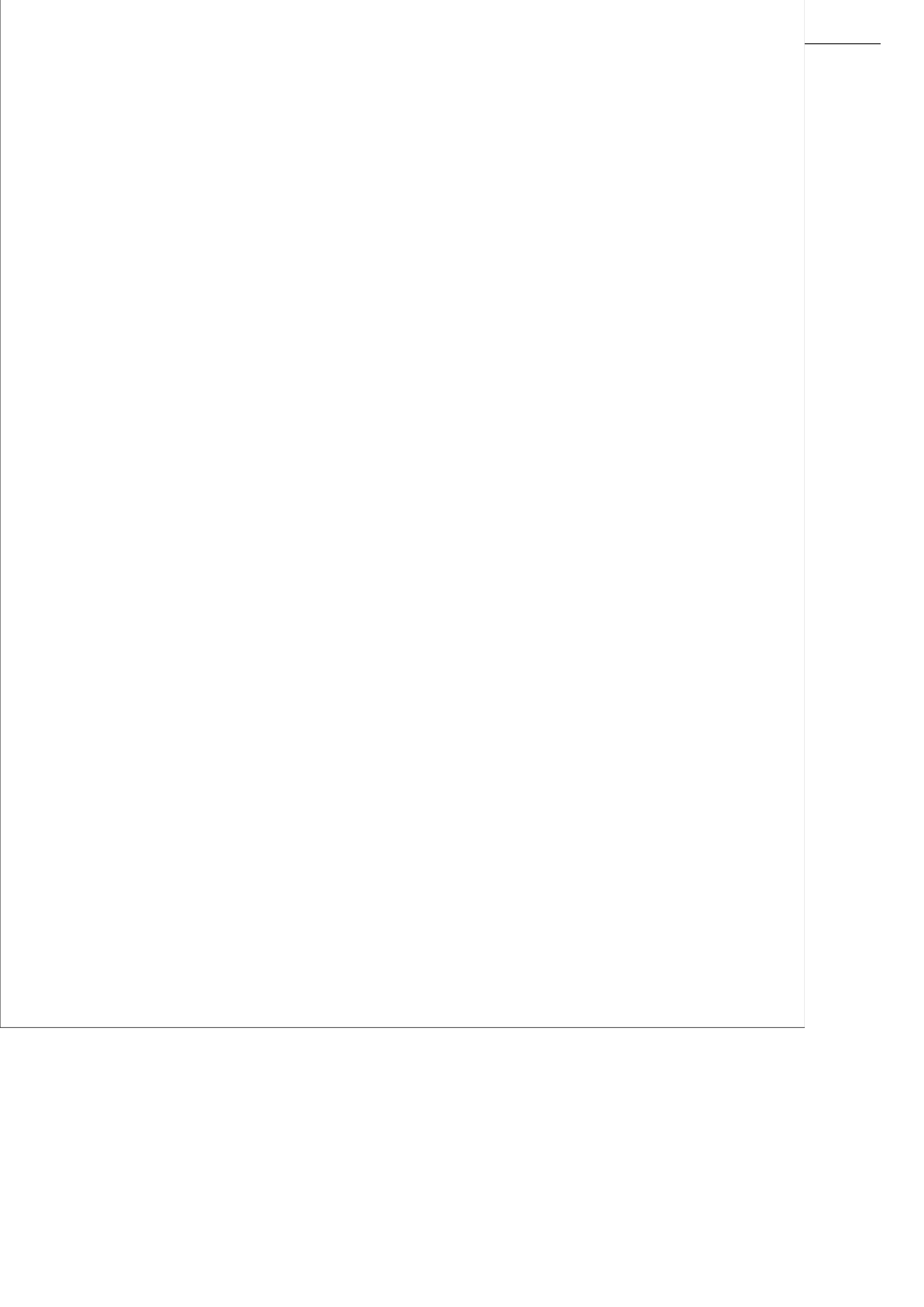


A Memoryless Edge Defence	648
The Edge-Corner Attack	650
Strategic and Tactical Stones	652
Corner Tactics	654
Defence on Large Square Boards	657
The 33×33 Board	658
The Centred King	658
Leaving the Central Region	658
The Cornered King	661
The Sidelined King	661
How Chas. Can Win on a 34×34 Board	663
Rectangular Boards	664
Many-Dimensional Angels	665
Games of Encirclement	665
Wolves-and-Sheep	665
Tablut	666
Saxon Hnefatafl	666
King and Rook Versus King	667
References and Further Reading	668
20 Fox and Geese	669
Some Properties of Our Strategy	672
What Is the Value of Fox-and-Geese?	673
Fox-Flocks-Fox	673
Towards Greater Precision	673
The Indefinite Board	674
How the Geese Survive Triumphantly	675
Eight Exciting Escapades	677
Tactics and Startegy for the Fox	681
Fox's Play on Definite Boards	686
The Scrimmage Sequence	687
Values of the Initial Positions	688
Sophisticated New Software	690
Values of FOXTAC Positions	692
High Region	692
Welton's Region of Value 2over	693
Lower "Delta" Region	697
Initial Values	697
Parity in the High Scrimmage Region	697
Early Values	700
The <i>abc</i> 's to Trifurcate the Two-Ish Transition	701
Extending the Backbone to Lower Altitudes	702
Early Values of Other Formations	702
Resolving the Exceptions	704

Museums	705
Solutions to Problems	709
Open Problems	710
Maharajah and Sepoys	710
References and Further Reading	710
21 Hare and Hounds	711
The French Military Hunt	711
Two Trial Games	713
History	713
The Different Kinds of Place	713
The Opposition	714
When Has the Hare Escaped?	715
Losing the Opposition	716
A Strategy for the Hare	718
On the Small Board	721
On the Medium and Larger Boards	722
Answers to Questions	725
A Sound Bound for a Hound?	725
All Is Found for the Small Board Hound	725
Proof of the Thirty-One Theorem	729
References and Further Reading	729
22 Lines and Squares	731
Tit-Tat-Toe, My First Go, Three Jolly Butcher Boys All in a Row	731
Magic Fifteen	732
Spit Not So, Fat Fop, as if in Pan!	732
Jam	732
How Long Can You Fool Your Friends?	733
Analysis of Tic-Tac-Toe	733
Ovid's Game, Hopscotch, Les Pendus	736
Six Men's Morris	737
Nine Men's Morris	737
Three Up	737
Four-in-a-Row	737
Five-in-a-Row	738
Go-Moku	740
Six, Seven, Eight, Nine, . . . , in a Row	740
n -Dimensional k -in-a-Row	742
Strategy Stealing in Tic-Tac-Toe Games	743
Hex	744
Bridgit	744
How Does the First Player Win?	744
The Shannon Switching Game	744



The Black Path Game	746
Lewthwaite's Game	747
Meander	747
Winners and Losers	748
Dodgem	749
Dodgerydoo	751
Philosopher's Football	752
Count foxy words And stay awake Using lively wit	756
Amazons	756
Checkers	757
Chess	757
Cherries	757
Clobber	757
Go	758
Konane	759
Reversi-Othello	760
Scrabble	760
Shogi	760
Sowing Games	761
Answers to Problems	761
References and Further Reading	762
Index to Volumes 1–3	770



Preface to the Second Edition

In the first edition of *Winning Ways*, which appeared in 1982, we were able to make a rather sharp distinction between those games in Part I, to which the major theory of addition applied directly, and those games in Part 3, which seemed to require more specialized techniques. However, subsequent research by an increasingly large community of combinatorial game theorists has begun to blur this distinction. We now have many more games whose strategies depend both on the general theory of Volume 1 as well as on more specialized results. Introductions to many of these games and some illustrative problems have been added to this new edition. Those that did not readily fit elsewhere can be found in the new Extras to Chapter 22 at the end of this volume. This volume also includes a major revision of the original Chapter 20 on the game of Fox and Geese. Its enhanced variation, Fox-Flocks-Fox, provides compelling illustrations of some of the challenging problems that can now be solved by appropriately combining theories from Volumes 1, 2, and 3 with innovative computing algorithms.

This new edition owes much to the supportive efforts of numerous friends and colleagues, including Noam Elkies, Tom Ferguson, Aviezri Fraenkel, Martin Gardner, Sol Golomb, Al Hales, Greg Kuperberg, Silvio Levy, Donald Knuth, Martin Kutz, Greg Martin, Victor Meally, Richard Nowakowski, Hilarie Orman, Marc Paulhus, Ed Pegg, Michael Reid, Thea van Roode, Katherine Scott, George Sicherman, Aaron Siegel, Neil Sloane, Sally Smith, William Spight, John Tromp, Jonathan Welton, Julian West, David Wilson, and David Wolfe, and to the very professional yet kindly support of our publishers, Alice and Klaus Peters.

Elwyn Berlekamp, University of California, Berkeley
John Conway, Princeton University
Richard Guy, The University of Calgary, Canada

June 23, 2003

Preface to the Original Edition

Does a book need a Preface? What more, after fifteen years of toil, do three talented authors have to add.

We can reassure the bookstore browser, “Yes, this is just the book you want!”

We can direct you, if you want to know quickly what’s in the book, to page xx. This in turn directs you to volumes 1,2,3 and 4.

We can supply the reviewer, faced with the task of ploughing through nearly a thousand information-packed pages, with some pithy criticisms by indicating the horns of the polylemma the book finds itself on. It is not an encyclopedia. It is encyclopedic, but there are still too many games missing for it to claim to be complete. It is not a book on recreational mathematics because there’s too much serious mathematics in it. On the other hand, for us, as for our predecessors Rouse Ball, Dudeney, Martin Gardner, Kraitchik, Sam Loyd, Lucas, Tom O’Beirne and Fred. Schuh, mathematics itself is a recreation. It is not an undergraduate text, since the exercises are not set out in an orderly fashion, with the easy ones at the beginning. They are there though, and with the hundred and sixty-three mistakes we’ve left in, provide plenty of opportunity for reader participation. So don’t just stand back and admire it, work of art though it is. It is not a graduate text, since it’s too expensive and contains far more than any graduate student can be expected to learn. But it does carry you to the frontiers of research in combinatorial game theory and the many unsolved problems will stimulate further discoveries.

We thank Patrick Browne for our title. This exercised us for quite a time. One morning, while walking to the university, John and Richard came up with “Whose game?” but realized they couldn’t spell it (there are three tooze in English) so it became a one-line joke on line one of the text. There isn’t room to explain all the jokes, not even the fifty-nine private ones (each of our birthdays appears more than once in the book).

Omar started as a joke, but soon materialized as Kimberly King. Louise Guy also helped with proof-reading, but her greater contribution was the hospitality which enabled the three of us to work together on several occasions. Louise also did technical typing after many drafts had been made by Karen McDermid and Betty Teare.

Our thanks for many contributions to content may be measured by the number of names in the index. To do real justice would take too much space. Here’s an abridged list of helpers: Richard Austin, Clive Bach, John Beasley, Aviezri Fraenkel, David Fremlin, Solomon Golomb, Steve Grantham, Mike Guy, Dean Hickerson, Hendrick Lenstra, Richard Nowakowski, Anne Scott, David Seal, John Selfridge, Cedric Smith and Steve Tschantz.

No small part of the reason for the assured success of the book is owed to the well-informed and sympathetic guidance of Len Cegielka and the willingness of the staff of Academic Press and of Page Bros. to adapt to the idiosyncrasies of the authors, who grasped every opportunity to modify grammar, strain semantics, pervert punctuation, alter orthography, tamper with traditional typography and commit outrageous puns and inside jokes.

Thanks also to the the Isaak Walton Killam Foundation for Richard's Resident Fellowship at The University of Calgary during the compilation of a critical draft, and to the National (Science & Engineering) Research Council of Canada for a grant which enabled Elwyn and John to visit him more frequently than our widely scattered habitats would normally allow.

And thank you, Simon!

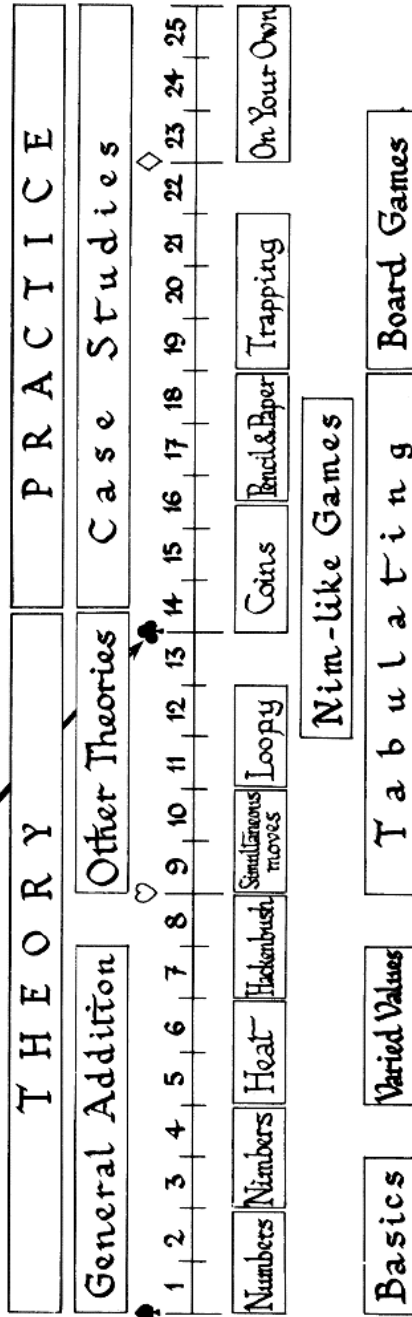
University of California, Berkeley, CA 94720 Elwyn Berlekamp
University of Cambridge, England, CB2 1SB John Conway
University of Calgary, Canada, T2N 1N4 Richard Guy

You are
now here

If you want to know roughly what's elsewhere,
turn to the little notes about our four main themes:

- Adding Games ... ♣ ... page 1
- Bending the Rules ... ♡ ... page 277
- Case Studies ... ♣ ... page 461
- Doing It Yourself ... ♢ ... page 803

There are a number of other connexions between various chapters of the book:



However, you should be able to pick any chapter and read almost all of it
without reference to anything earlier, except perhaps the basic ideas at the start of the book.



Games in Clubs!

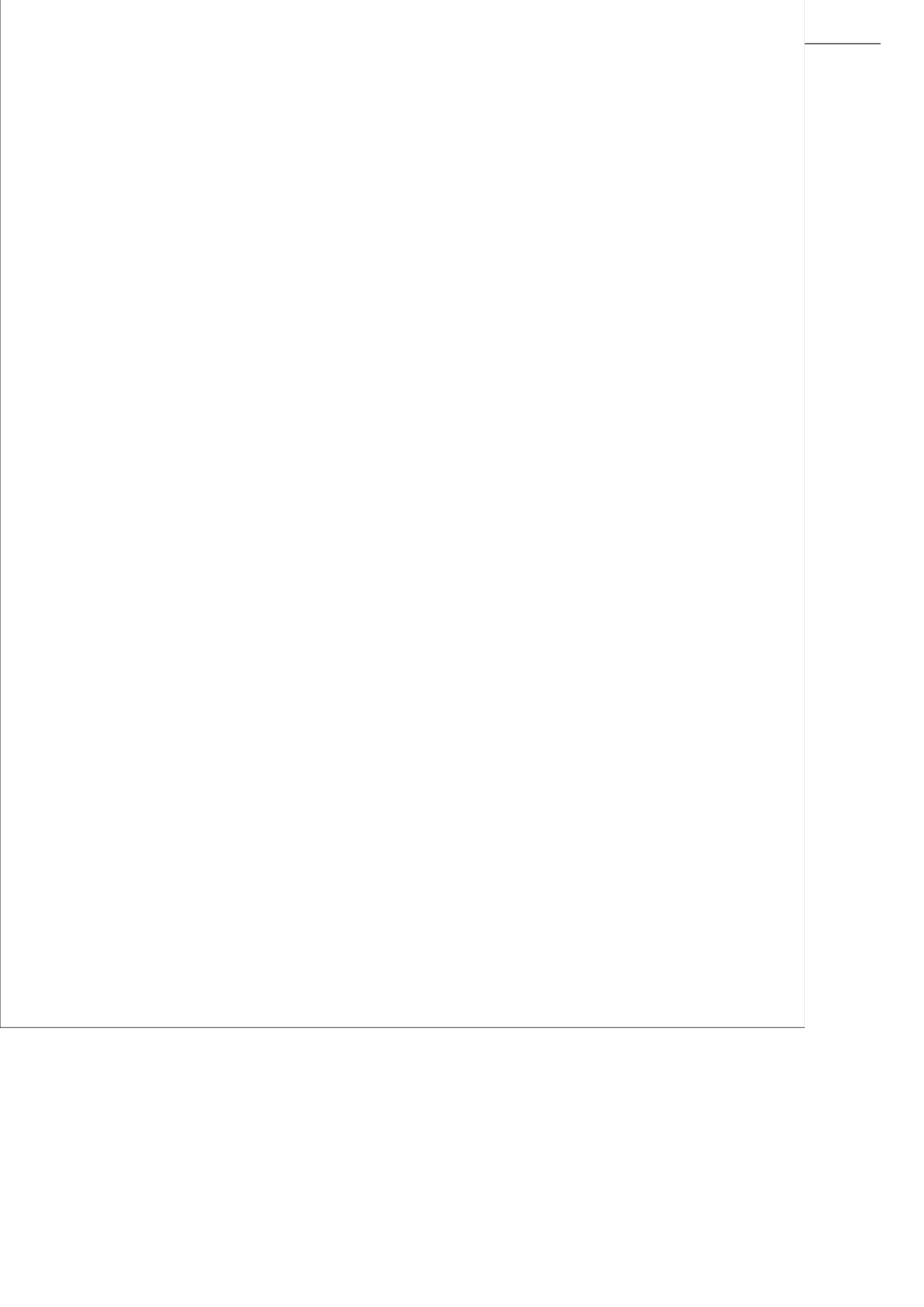
To be an Englishman is to belong
to the most exclusive club there is.
Ogden Nash, *England Expects*.

There are lots of games for which the theories we've now developed are useful, and even more for which they're not, and we've grouped them into clubs according to how you play them.

First some games you can play with coins, either by turning them over (Chapter 14) or moving them along strips or about in heaps (Chapter 15).

Then games for which you'll need pencil and paper, perhaps to draw straight lines (Chapter 16), or curved ones (Chapter 17) or merely to do the calculations in Chapter 18.

And for board games we have three case studies in which one player wins by trapping his opponent (Chapters 19, 20, 21) and finally many more which are usually won by the first player to establish some kind of winning configuration (Chapter 22).



Turn and Turn About

Because I do not hope to turn again
Because I do not hope
Because I do not hope to turn.

T. S. Eliot, *Ash Wednesday*, I.

Open not thine heart to every man, lest he requite thee
with a shrewd turn.

Ecclesiasticus, 8:19.

These games, based on an idea of H. W. Lenstra, are similar in that they all involve turning things over, but we shall see that they call for a variety of strategies.

Turning Turtles

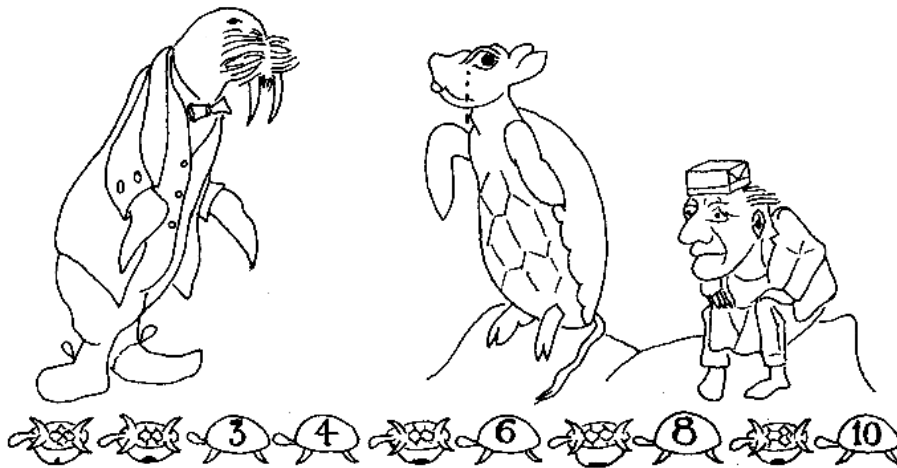


Figure 1. Playing Turning Turtles.

In Fig. 1 the Walrus and the Carpenter are playing a rather cruel game. At each move a player must put one turtle on its back and may also turn over any single turtle to the left of it. This second turtle, unlike the first, may be turned either onto its feet or onto its back. The player wins who turns the last turtle upside-down. Which turtles should the Walrus (*l.*) turn?

Like most readers of this book, he wearily suspects another disguise for Nim. Here only turtles 3, 4, 6, 8 and 10 are on their feet, and since the nim-sum of 3, 4 and 6 is 1, he may turn 10 onto its back and 9 onto its feet, producing 3, 4, 6, 8, 9, a \mathcal{P} -position since $8 \oplus 9 = 1$. The Carpenter (*r.*) responds by turning 8 and 5 producing the position 3, 4, 5, 6, 9 as in Fig. 2.



Figure 2. After the Carpenter's Reply.

In Nim there is only one good move from this position—reduce 9 to 4, so as to produce, 3, 4, 4, 5, 6, which, since two equal Nim heaps may be cancelled, is much the same as 3, 5, 6, which the Walrus reaches by turning both 9 and 4 on their backs (Fig. 3).

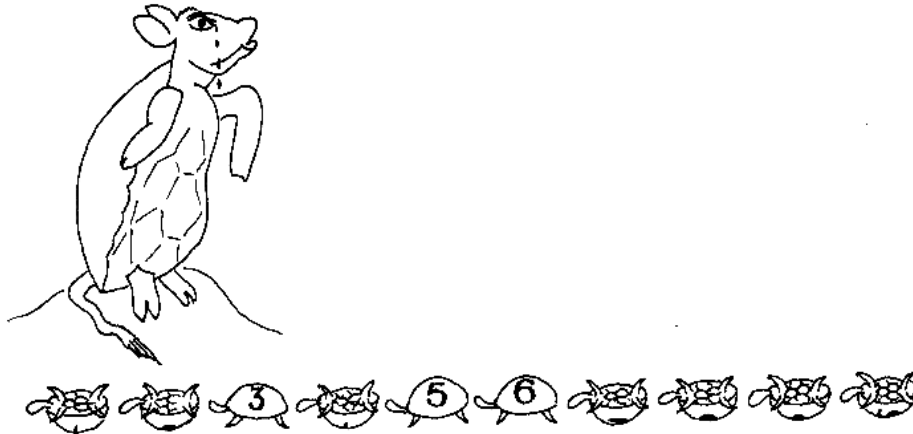


Figure 3. How the Walrus Won.

Nim moves become turtle turns as follows. We reduce a heap to a size not already present by turning one turtle on its back and putting another on its feet, as in the Walrus's opening move. If a heap of the reduced size is already present, we turn two turtles on their backs as in the Walrus's response to the Carpenter's move (cancelling two equal heaps). To eliminate a heap entirely, we merely turn the appropriate turtle. So since 4, 6, 8, 10 is a \mathcal{P} -position, the Walrus could have won from Fig. 1 by just turning turtle 3.

Since all our turning games are impartial, they are solved by computing the nim-values, and often may be thought of as heap games in disguise; but many games with interesting theories are more naturally suggested by the turning version.

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